

Solutions to
2026 AP Calculus BC
Free Response Questions

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1. Unless otherwise indicated, values of $M(t)$ used in the following are obtained from the table given in the problem.

A. The average rate of change of $M(t)$ over the interval $5 \leq t \leq 10$ is, in birds per day per day,

$$\frac{M(10) - M(5)}{10 - 5} = \frac{16 - 7}{5} = \frac{9}{5}. \quad (1)$$

B. i. The midpoint sum for $\int_0^{30} M(t) dt$ with the three subintervals $[0, 10]$, $[10, 20]$, and $[20, 30]$ is

$$M(5)(10 - 0) + M(15)(20 - 10) + M(25)(30 - 20) \quad (2)$$

$$= 7 \cdot 10 + 6 \cdot 10 + 2 \cdot 10 \quad (3)$$

$$= 70 + 60 + 20 \quad (4)$$

$$= 150. \quad (5)$$

ii. $M(t_0)$ denotes the rate at which male birds arrive at time t_0 , so $\int_0^{30} M(t) dt$ gives the total number of birds that arrive during the thirty-day period $0 \leq t \leq 30$.

C. We are given

$$F(t) = \begin{cases} 0 & \text{for } 0 \leq t < 15 \\ 18 + 16 \sin\left(\frac{\pi}{20}(t + 15)\right) & \text{for } 15 \leq t \leq 45 \end{cases} \quad (6)$$

as the rate at which female birds arrive during the period $15 \leq t \leq 45$. Thus, during the period $15 \leq t \leq 45$, the total number of female birds that arrive is

$$\int_{15}^{45} F(t) dt = \int_{15}^{45} \left[18 + 16 \sin \left(\frac{\pi}{20}(t + 15) \right) \right] dt \quad (7)$$

$$(8)$$

Numeric integration gives

$$\int_{15}^{45} F(t) dt \sim 641.859, \quad (9)$$

or, to the nearest integer, 642.

- D. We easily calculate that $D(15) = 4$, while $D(20) \sim -1.686$. The function M is given differentiable, and the function F is surely differentiable. Both are therefore continuous, and $D(t) = M(t) - F(t)$ is therefore continuous. The Intermediate Value Theorem for Continuous Functions therefore guarantees that there is a value ξ in the interval $15 < t < 20$ where $D(\xi) = 0$.

2. A. The area, A , of a polar curve $r = r(\theta)$, where $\alpha \leq \theta \leq \beta$, is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \quad (10)$$

We are given $r(\theta) = 3 + 2 \sin 2\theta + \cos 2\theta$, with $0 \leq \theta \leq \pi$, so our desired area is

$$A = \frac{1}{2} \int_0^{\pi} (3 + 2 \sin 2\theta + \cos 2\theta)^2 d\theta. \quad (11)$$

Integrating numerically, we find that

$$A \sim 18.064. \quad (12)$$

- B. We know that

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}. \quad (13)$$

We are given that the slope of the tangent to the curve, i.e., the value of $\frac{dy}{dx}$, at a certain point is $-\frac{3}{7}$ —that is $\frac{dy}{dx} = -\frac{3}{7}$. We are also given $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$, so

$$-\frac{3}{7} = \frac{\left(\frac{3\sqrt{2}}{2}\right)}{\left(\frac{dx}{d\theta}\right)}, \quad (14)$$

whence

$$\frac{dx}{d\theta} = -\frac{3\sqrt{2}}{2} \cdot \frac{7}{3} = -\frac{7\sqrt{2}}{2}. \quad (15)$$

- C. i. Critical points are points interior to the interval of interest where the derivative vanishes or is undefined. Here, we know that

$$r'(\theta) = 4 \cos 2\theta - 2 \sin 2\theta, \quad (16)$$

so that $r'(\theta)$ is meaningful for all values of θ and r' is continuous everywhere. We therefore seek points in $(0, \pi/2)$ where $r'(\theta) = 0$ or where

$$4 \cos 2\theta - 2 \sin 2\theta = 0. \quad (17)$$

Equivalently,

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = 2. \quad (18)$$

Consequently, the only critical point in $(0, \pi/2)$ lies at $\theta_0 := \frac{1}{2} \arctan 2$. We have, approximately, $\theta_0 \sim 0.55357 \sim 0.554$.

ii. We have

$$r''(\theta) = \frac{d}{d\theta} [4 \cos 2\theta - 2 \sin 2\theta] \quad (19)$$

$$= -4 \cos 2\theta - 8 \sin 2\theta. \quad (20)$$

Now, $0 < \frac{1}{2} \arctan 2 < \frac{\pi}{4} \sim 0.785$, so $2\theta_0$ lies well within the first quadrant, where the sine and the cosine are both positive. Thus, $r''(\theta_0) < 0$, and the Second Derivative Test assures us that $r(\theta)$ has a relative maximum at $\theta = \theta_0$.

Alternate Solution: We have $\cos 2\theta > -0$ on $(0, \pi/4)$ and $r'(0) = 4 > 0$. Moreover, the continuous r' has the Intermediate Value Property, meaning that the only point in $(0, \pi/4)$ where $r'(\theta)$ can change sign is the critical point $\theta = \theta_0$. Thus when $0 \leq \theta < \theta_0$, we have $r'(\theta) \cos 2\theta > 0$.

A similar argument, based on the fact that $r'(\pi/4) < 0$, establishes that $r'(\theta) < 0$ when $\theta_0 < \theta < \frac{\pi}{4}$.

It now follows from the First Derivative Test that $r(\theta)$ has a relative maximum at $\theta = \theta_0$.

D. The quantity $r(\theta)$ itself being the distance from the origin to the point $(r(\theta), \theta)$, the average distance, \bar{r} , from the origin to the point on the polar curve on the

interval $\frac{\pi}{2} \leq \theta \leq \pi$ is

$$\bar{r} = \frac{2}{\pi} \int_{\pi/2}^{\pi} r(\theta) d\theta \quad (21)$$

$$= \frac{2}{\pi} \int_{\pi/2}^{\pi} [3 + 2 \sin 2\theta + \cos 2\theta] d\theta \quad (22)$$

$$= \frac{2}{\pi} \left(3\theta + \frac{1}{2} \sin 2\theta - \cos 2\theta \right) \Big|_{\pi/2}^{\pi} \quad (23)$$

$$= \frac{2}{\pi} \left[\left(3\pi - 1 \right) - \left(\frac{3\pi}{2} + 1 \right) \right] \quad (24)$$

$$= \frac{2}{\pi} \left[\frac{3\pi}{2} - 2 \right] \sim 1.72676 \sim 1.727. \quad (25)$$

3. A. The given slope field can't be the slope field for the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20) \quad (26)$$

because when $H > 20$ the equation (26) tells us that $\frac{dH}{dt}$ is negative, while all the slopes that appear in the given slope field are positive—even though $H > 20$ throughout the region shown.

- B. We are given $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$ and $H(0) = 75$. Thus, when $t = 0$ we have

$$H'(0) = -\frac{1}{15}[H(0) - 20] \quad (27)$$

$$= -\frac{1}{15}(75 - 20) \quad (28)$$

$$= -\frac{55}{15} = -\frac{11}{3}. \quad (29)$$

The slope of the line tangent to the graph of H at time $t = 0$ is $H'(0) = -\frac{11}{3}$.

- C. If (26) holds, if $H'' = \frac{1}{225}(H - 20)$, and if the linearization of H at $t = 0$ is used to approximate $H(5)$, we expect that the approximation will underestimate $H(5)$. This is because we know that $H(0) = 75$, so that $H''(0) = \frac{1}{225}[H(0) - 20] = \frac{11}{45} > 0$, making the curve concave upward near $t = 0$. Thus, the tangent line through $(0, H(0))$ lies below the $H(t)$ curve in that region.

Note: I believe this to be the argument the readers expect to see. It's a shaky one, because we have no information about how rapidly $H''(t)$ changes, and the interval over which we make the linearization approximation is fairly large. From what we know at this point, we should consider the possibility that the second derivative becomes negative, so that the curve turns downward—possibly even crossing the tangent line. The only way I see to do this is to use the solution that we find in the next part of the problem—and we can then calculate directly whether the linearization gives an over estimate or an underestimate. This seems to me to be a poorly written problem.

- D. If $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$ and $H(0) = 75$, then

$$\frac{dH}{H - 20} = -\frac{dt}{15}, \quad (30)$$

so that

$$\int_{75}^H \frac{dh}{h-20} = -\frac{1}{15} \int_0^t d\tau \quad (31)$$

$$\ln(h-20) \Big|_{75}^H = -\frac{1}{15} \tau \Big|_0^t \quad (32)$$

$$\frac{H-20}{55} = e^{-t/15} \quad (33)$$

$$H = 20 + 55e^{-t/15}. \quad (34)$$

4. A. If $g(x) = f(x) - \ln x$ for $x > 0$, then

$$g'(x) = f'(x) - \frac{1}{x}, \quad (35)$$

so that

$$g'(2) = f'(2) - \frac{1}{2} = 1.5 - \frac{1}{2} = 1, \quad (36)$$

it being given that $f'(2) = 1.5$.

- B. The graph of f has a point of inflection at every point where f' has a local extremum—that is, at points where the monotonicity of f' changes. From the given graph, we find such points at $x = -3$, at $x = 1$, and at $x = 3$.
- C. The graph of f is increasing on intervals where $f'(x) > 0$, decreasing on intervals where $f'(x) < 0$. The graph of f is concave upward on intervals where $f'(x)$ is increasing, concave downward on intervals where $f'(x)$ is decreasing. From the graph of f' , we see that f decreasing and concave downward on $(-4, -3)$, decreasing and concave upward on $(-3, -2)$, increasing and concave upward on $(-2, 1)$ and on $(3, 4)$, and *increasing but concave downward* on $(1, 3)$.
- D. The function f has a critical point at $x = -2$ where $f'(x)$ is negative to the left but positive to the right—making this point a relative minimum. It has another critical point at $x = 3$, where $f'(x)$ is positive to both sides—making this point a saddle point, which can be neither a minimum nor a maximum. There can be no minimum at the left endpoint, $x = -4$, because $f'(x) < 0$ for values of x immediately to the right of $x = -4$ —meaning that f decreases just to the right of $x = -4$. Similarly, there can be no minimum at the right endpoint $x = 4$, because $f'(x) > 0$ for values of x immediately to the left $x = 4$. The absolute minimum guaranteed by the Extreme Value Theorem (which applies because f is differentiable and therefore continuous) must occur at either a critical point or an end point, so the absolute minimum value of $f(x)$ on $[-4, 4]$ occurs at $x = -2$.

The absolute maximum guaranteed by the Extreme Value Function must occur at one of the endpoints, $x = -4$ or $x = 4$, because all of the critical points are accounted for.

We have, by the Fundamental Theorem of Calculus,

$$f(4) = f(-4) + \int_{-4}^4 f'(x) dx. \quad (37)$$

Let g be the function defined on $[-4, 4]$ as the function whose graph is the union of the line segment connecting the point $(-4, -1)$ and the point $(-2, -1)$,

the line segment connecting the point $(-2, 0)$ and the point $(1, 3)$, the open line segment connecting the point $(1, 3)$ and the point $(2, 0)$, and, finally, the open-closed line segment connecting the point $(2, 0)$ with the point $(4, 0)$. More formally,

$$g(x) = \begin{cases} -1 & -4 \leq x < -2 \\ 2 + x & -2 \leq x \leq 1 \\ 6 - 3x & 1 < x < 2 \\ 0 & 2 \leq x \leq 4 \end{cases} . \quad (38)$$

It is clear from the graph that $f'(x) \geq g(x)$ for all $x \in [-4, 4]$. Consequently,

$$f(4) = f(-4) + \int_{-4}^4 f'(x) dx \quad (39)$$

$$\geq f(-4) + \int_{-4}^4 g(x) dx \quad (40)$$

$$\geq f(-4) + \int_{-4}^{-2} (-1) dx + \int_{-2}^1 (2 + x) dx + \int_1^2 (6 - 3x) dx + \int_2^4 0 dx \quad (41)$$

$$\geq f(-4) + (-2) + \frac{1}{2} \cdot 3 \cdot 3 + \frac{1}{2} \cdot 1 \cdot 3 + 0 \quad (42)$$

$$\geq f(-4) + 4. \quad (43)$$

Therefore the absolute maximum for f must be at $x = 4$, because we have shown that $f(4) > f(-4)$.

5. We are given $f(x) = \sqrt[3]{x-1}$ and $g(x) = e^{(-2x+4)}$.

A.

$$\int_1^2 f(x) dx = \int_1^2 \sqrt[3]{x-1} dx. \quad (44)$$

We let $v = x - 1$, so that $dv = dx$. Moreover, $v = 0$ when $x = 1$ and, $v = 1$ when $x = 2$. Thus,

$$\int_1^2 \sqrt[3]{x-1} dx = \int_0^1 v^{1/3} dv \quad (45)$$

$$= \frac{3}{4} v^{4/3} \Big|_0^1 = \frac{3}{4} \cdot 1^{4/3} - \frac{3}{4} \cdot 0^{4/3} = \frac{3}{4}. \quad (46)$$

B. The required integral (using the disk method) is $\pi \int_1^2 (x-1)^{2/3} dx$. Alternately, the shell method gives $2\pi \int_0^1 y(1-y^3) dy$. Evaluation, although not required, of either integral yields $\frac{5\pi}{3}$.

C. The length, s , of the curve $y = f(x)$, $1 \leq x \leq 2$ is given by

$$s = \int_1^2 \sqrt{1 + [f'(x)]^2} dx \quad (47)$$

$$= \int_1^2 \sqrt{1 + \frac{1}{9(x-1)^{4/3}}} dx \quad (48)$$

The other two sides of the region R both have length 1, so P_R , the perimeter of R is given by

$$P_R = 2 + \int_1^2 \sqrt{1 + \frac{1}{9(x-1)^{4/3}}} dx. \quad (49)$$

Alternate Solution: The length of the curve can also be found by writing its equation as $k(y) = 1 + y^3$ and calculating

$$s = \int_0^1 \sqrt{1 + [k'(y)]^2} dy = \int_0^1 \sqrt{1 + 9y^4} dy. \quad (50)$$

Neither this integral nor the one in the first solution is elementary.

D. We have

$$\int_2^{\infty} g(x) dx = \lim_{T \rightarrow \infty} \int_2^T e^{-2x+4} dx. \quad (51)$$

Let $w = -2x + 4$, so that $dw = -2 dx$, or $dx = -\frac{dw}{2}$. If $x = 2$ then $w = 0$, and if $x = T$, then $w = 4 - 2T$. Thus,

$$\int_2^{\infty} e^{-2x+4} dx = \lim_{T \rightarrow \infty} \int_2^T e^{-2x+4} dx \quad (52)$$

$$= -\frac{1}{2} \lim_{T \rightarrow \infty} \int_0^{4-2T} e^w dw \quad (53)$$

$$= -\frac{1}{2} \lim_{T \rightarrow \infty} e^w \Big|_0^{4-2T} \quad (54)$$

$$= -\frac{1}{2} \lim_{T \rightarrow \infty} (e^4 \cdot e^{-2T} - 1) \quad (55)$$

$$= -\frac{1}{2} \left(\lim_{T \rightarrow \infty} e^4 \cdot \lim_{T \rightarrow \infty} e^{-2T} - 1 \right) = \frac{1}{2}. \quad (56)$$

6. We are given

$$g(x) = \sum_{n=0}^{\infty} 2 \left(\frac{-x}{5} \right)^n, \quad (57)$$

where the series converges—which we are also given is for $|x| < 5$.

A.

$$g(3) = \sum_{n=0}^{\infty} 2 \left(\frac{-3}{5} \right)^n. \quad (58)$$

This is a geometric series of the form $\sum_{n=0}^{\infty} ar^n = a \sum_{n=0}^{\infty} r^n$, which we know converges and satisfies

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad (59)$$

when $|r| < 1$. Here, $r = -\frac{3}{5}$, so that $|r| = \frac{3}{5} < 1$, and

$$g(3) = \frac{2}{\left(1 + \frac{3}{5}\right)} = 2 \cdot \frac{5}{8} = \frac{5}{4}. \quad (60)$$

B. We obtain the Maclaurin series for $f(x) = g'(x)$ by differentiating, term by term, the Maclaurin series for g .

$$g(x) = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 + \dots, \quad (61)$$

so

$$f(x) = g'(x) = -\frac{2}{5} + \frac{4}{25}x - \frac{6}{125}x^2 + \frac{8}{625}x^3 + \dots \quad (62)$$

C. We begin by noting that if $|x| < 5$, then whatever the integer $n > 0$ may be, we have

$$\left| 2 \left(\frac{-x}{5} \right)^{n+1} \right| / \left| 2 \left(\frac{-x}{5} \right)^n \right| = \frac{|x|}{5} < 1. \quad (63)$$

Consequently, for such values of x , $\left| 2 \left(\frac{-x}{5} \right)^n \right|$ decreases monotonically, and, because the series converges for such x , the limit is zero as $n \rightarrow \infty$. We may,

according to the Alternating Series Test, therefore estimate the error in using its quadratic Maclaurin polynomial to approximate $f(x)$ by evaluating the magnitude of the cubic term, $\left|\frac{8}{625}x^3\right|$, of the Maclaurin series. We conclude that

$$\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| < \frac{8}{625} \left(\frac{5}{2}\right)^3 = \frac{1}{5}. \quad (64)$$

Alternate Solution: Using (59), we know that

$$g(x) = \frac{2}{\left(1 + \frac{x}{5}\right)} = \frac{10}{5 + x}. \quad (65)$$

Hence

$$f(x) = g'(x) = 10 \frac{d}{dx}(5 + x)^{-1} = -\frac{10}{(5 + x)^2}, \quad (66)$$

and

$$f\left(\frac{5}{2}\right) = -\frac{10}{\left(5 + \frac{5}{2}\right)^2} = -\frac{10}{\left(\frac{15}{2}\right)^2} \quad (67)$$

$$= -\frac{8}{45} \quad (68)$$

Thus,

$$\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| = \left|-\frac{8}{45} + \frac{3}{10}\right| \quad (69)$$

$$= \frac{11}{90} \quad (70)$$

$$< \frac{18}{90} = \frac{1}{5}. \quad (71)$$

D. If $h(x) = 25g(x) - 2e^x$ then

- i. the first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2}$.
- ii. The first three terms of the Maclaurin series for $25g(x)$ are $50 - 10x + 2x^2$. We combine this fact with the fact adduced in the previous part of this problem to find that the first three non-zero terms of the Maclaurin series for $25g(x) - 2e^x$ are $48 - 12x + x^2$.

Alternate Solution: Taking $g(x) = \frac{10}{5+x}$ as previously calculated, we see that

$$h(x) = \frac{250}{5+x} - 2e^x. \quad (72)$$

Thus

$$h(0) = 48 \quad (73)$$

$$h'(x) = -\frac{250}{(5+x)^2} - 2e^x \Rightarrow h'(0) = -\frac{250}{25} - 2e^0 = -12 \quad (74)$$

$$h''(x) = \frac{500}{(5+x)^3} - 2e^x \Rightarrow h''(0) = 2 \quad (75)$$

Thus, the first three non-zero terms in the Maclaurin series for $h(x)$ are

$$h(0) + h'(0)x + \frac{h''(0)}{2}x^2 = 48 - 12x + \frac{2}{2}x^2 \quad (76)$$

$$= 48 - 12x + x^2. \quad (77)$$