AP Calculus 2005 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The general acceleration vector $\mathbf{a}(t)$ at time t is given by

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt}\left\langle x'(t), y'(t)\right\rangle && (1)\\
&= \frac{d}{dt}\left\langle 12t - 3t^2, \ln[1+(t-4)^4]\right\rangle && (2)\\
&= \left\langle 12 - 6t, \frac{4(t-4)^3}{1+(t-4)^4}\right\rangle. && (3)
\end{aligned}$$

Thus,

$$\mathbf{a}(2) = \left\langle 0, -\frac{32}{17}\right\rangle. \quad (4)$$

Speed $\sigma(t)$ at time t is given by

$$\begin{aligned}
\sigma(t) &= \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)} = \sqrt{\langle x'(t), y'(t)\rangle \cdot \langle x'(t), y'(t)\rangle} = \sqrt{[x'(t)]^2 + [y'(t)]^2} && (5)\\
&= \sqrt{(12t - 3t^2)^2 + (\ln[1+(t-4)^4])^2}, && (6)
\end{aligned}$$

so that $\sigma(2) = \sqrt{144 + (\ln 17)^2} \sim 12.32993$.

1.2 Part b

The y-coordinate of P, the object's position when $t = 2$ is, by the Fundamental Theorem of Calculus, given by

$$y(2) = 5 + \int_0^2 \ln\left[1 + (t-4)^4\right]\, dt \sim 13.67145, \tag{7}$$

where it has been necessary to carry out the non-elementary integration numerically.

1.3 Part c

The tangent line at $t = 2$ is parallel to the vector $\langle x'(2), y'(2)\rangle = \langle 12, \ln 17\rangle$, and thus has slope $\frac{1}{12}\ln 17$. An equation is thus

$$y = \left[5 + \int_0^2 \ln\left[1 + (t-4)^4\right]\, dt\right] + \left[\frac{1}{12}\ln 17\right](x-3), \tag{8}$$

or, approximately

$$y = 12.96315 + 0.23610x \tag{9}$$

2 Problem 2

2.1 Part a

At time $t = 15$, water is entering the tank at the rate of

$$W(15) = 95\sqrt{15}\sin^3\frac{15}{6} \sim 131.78231 \text{ gallons per hour,} \tag{10}$$

and is being removed from the tank at the rate of

$$R(15) = 275\sin^2\frac{15}{3} \sim 252.87234 \text{ gallons per hour.} \tag{11}$$

When $t = 15$, the removal rate is larger than the supply rate, so the amount of water in the tank is decreasing when $t = 15$.

2.2 Part b

The amount, $A(t)$, of water in the tank at time t s, by the Fundamental Theorem of Calculus,

$$A(t) = 1200 + \int_0^t \left[W(\tau) - R(\tau)\right], d\tau. \tag{12}$$

Thus, integrating numerically, we find that

$$A(18) \sim 1309.78818 \text{ gallons.} \tag{13}$$

So, to the nearest whole number, there are 1310 gallons of water in the tank at $t = 18$.

2.3 Part c

We seek the zeros of $A'(t) = W(t) - R(t)$ in the interval $(0, 18)$. Solving numerically, we find that these are $t \sim 6.49484$ and $t \sim 12.9748$. Because $A'(t)$ is defined for all $t \in (0, 18)$, we know that the absolute minimum value of $A(t)$ for $t \in [0, 18]$ must be one of the four values $A(0) = 1200$, $A(6.49484) \sim 525.24215$, $A(12.97482) \sim 1697.44124$, and $A(18) \sim 1309.78818$. (We have used (12) to calculate all but the first of these four values, carrying out the required integrations numerically.) Thus, the absolute minimum amount of water in the tank during the time interval $[0, 18]$ occurs when $t \sim 6.49484$; that minimum value is about 525.24215 gallons.

2.4 Part d

With A and R as defined above, we must solve for k in the equation

$$A(18) - \int_{18}^{k} R(\tau)\, d\tau = 0. \tag{14}$$

Note: Solution of (14) is not required. For the curious, numerical techniques give $k \sim 29.19242$ for the solution.

3 Problem 3

3.1 Part a

We have $f'(0) = 0$, because the tangent line at $x = 0$ is given horizontal. From the formula given for the derivatives, we have

$$f''(0) = \frac{(-1)^3 \cdot 3!}{5^2 \cdot 1^2} = -\frac{6}{25} < 0. \tag{15}$$

By the Second Derivative Test, f has a local maximum at $x = 2$.

3.2 Part b

The third degree Taylor polynomial, $T_3(x)$ at $x = 0$ is

$$\begin{aligned} T_3(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 && (16) \\ &= 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3. && (17) \end{aligned}$$

3.3 Part c

If a_n is the coefficient of x^n in the Taylor series for f, then

$$a_n = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2 \cdot n!} = \frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}. \tag{18}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} &= \frac{|x|}{5} \lim_{n\to\infty} \left[\frac{(n-1)^2}{(n+1)} \cdot \frac{(n+2)}{n^2} \right] && (19) \\ &= \frac{|x|}{5} \lim_{n\to\infty} \frac{(1-1/n)^2(1+2/n)}{(1+1/n)} = \frac{|x|}{5}. && (20) \end{aligned}$$

We conclude, from the Ratio Test, that the series converges when $|x| < 5$ but diverges when $|x| > 5$. The series therefore has radius of convergence 5.

4 Problem 4

4.1 Part a

$g(-1) = \int_{-4}^{-1} f(t)\,dt$ is the negative of the area of the trapezoid defined by the x-axis, the vertical lines $x = -4$ and $x = -1$, and the line segment joining the points $(-4, -3)$ and $(-1, -2)$. Thus,

$$g(-1) = -\frac{1}{2} \cdot (3 + 2) \cdot 3 = -\frac{15}{2}. \tag{21}$$

By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(-1) = -2$. It also follows that $g''(x) = f'(x)$, if, and only if, the latter exists. Because of the corner in the graph of $f(x)$ at the point corresponding to $x = -1$, $f'(-1)$ does not exist. (In fact, $f'_{-}(-x) = 1/3$, while $f'_{+}(-1) = 2$.) Thus, $g''(-1)$ does not exist.

4.2 Part b

The inflection points of g occur where $g' = f$ has relative extrema. But f has just one relative extremum in the interval $(-4, 3)$, at $x = 1$—as is evident from the graph. Thus, the only inflection point for g is to be found at $x = 1$.

4.3 Part c

If

$$h(x) = \int_{x}^{3} f(t)\,dt = -\int_{3}^{x} f(t)\,dt. \tag{22}$$

then the zeros of h are to be found at those values of x for which the graph of f over the interval whose endpoints are 3 and x has just as much area above the horizontal coordinate axis as below. These values are evidently $x = -1$ and $x = 1$. And, of course, we shouldn't forget the trivial solution $x = 3$.

4.4 Part d

With h as given in Part c, above, we have, by the Fundamental Theorem of Calculus, $h'(x) = -f(x)$. Therefore, h is decreasing on (the closures of) those intervals for which $-f(x) < 0$, or, equivalently, where $f(x) > 0$. From the graph, it is thus evident that h is decreasing on $[0, 2]$.

5 Problem 5

5.1 Part a

On the curve $y^2 = 2 + xy$, we treat y as though it is (at least, near each point on the curve) a function of x. Then, differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[2 + xy\right] \tag{23}$$

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx} \tag{24}$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y \tag{25}$$

$$\frac{dy}{dx} = \frac{y}{2y - x}, \tag{26}$$

at least as long as $2y - x \neq 0$. But if $2y - x = 0$, then $x = 2y$, and the equation $y^2 = 2 + xy$ becomes $y^2 = 2 + (2y)y$, or $y^2 = -2$, which is not possible for real values of y. We conclude that (26) gives $\frac{dy}{dx}$ at all points of the curve.

5.2 Part b

If

$$\frac{1}{2} = y' = \frac{y}{2y - x}, \text{then} \tag{27}$$

$$2y - x = 2y, \tag{28}$$

and $x = 0$. Substituting this in the original equation, we find that $y^2 = 2$. The required points are therefore $(0, \sqrt{2})$ and $(0, -\sqrt{2})$.

5.3 Part c

If the tangent line to the curve $y^2 = 2 + xy$ is horizontal at a point (x_0, y_0), we must have

$$0 = y'(x_0) = \frac{y_0}{2y_0 - x_0}. \tag{29}$$

As we have seen in Part a, above, if (x_0, y_0) is a point on the curve, then $2y_0 + x_0 = 0$ is not possible, so (29) means that $y_0 = 0$. But then $0 = y_0^2 = 2 + x_0 y_0 = 2 + 0 = 2$, which means that $0 = 2$. The contradiction show that there can be no point on the curve $y^2 = 2 + xy$ where the tangent line is horizontal.

5.4 Part d

We differentiate the equation for the curve implicitly again, but this time we treat x and y both as functions of a third variable t, and we take the prime to mean differentiation with respect to t. We have

$$\frac{d}{dt}y^2 = \frac{d}{dt}\left[2 + xy\right]; \tag{30}$$

$$2y\frac{dy}{dy} = y\frac{dx}{dt} + x\frac{dy}{dt}, \tag{31}$$

or $2yy' = yx' + xy'$. Putting $y = 3$, $y' = 6$, in both the original equation and the derived equation leads to the system of equations

$$9 = 2 + 3x \tag{32}$$

$$36 = 3x' + 6x. \tag{33}$$

From the first of these two, we see that $x = 7/3$, and substituting this for x in the second equation yields $36 = 3x' + 14$, whence $x' = 22/3$.

6 Problem 6

6.1 Part a

The required area is

$$\int_0^k \frac{dx}{x+2} = \ln(k+2) - \ln 2. \tag{34}$$

6.2 Part b

The required volume is

$$\pi \int_0^k \frac{dx}{(x+2)^2} = -\frac{\pi}{x+2}\bigg|_0^k = \frac{\pi k}{2(k+2)}. \tag{35}$$

6.3 Part c

The volume of the solid generated by S is given by the improper integral

$$\pi \int_k^\infty \frac{dx}{(x+2)^2} = \pi \lim_{T\to\infty} \int_k^T \frac{dx}{(x+2)^2} \tag{36}$$

$$= -\pi \lim_{T\to\infty} \frac{1}{x+2}\bigg|_k^T \tag{37}$$

$$= -\pi \lim_{T\to\infty} \left[\frac{1}{T+2} - \frac{1}{k+2}\right] = \frac{\pi}{k+2}. \tag{38}$$

This volume is equal to that of the solid of Part b, above, if and only if

$$\frac{\pi k}{2(k+2)} = \frac{\pi}{k+2}, \tag{39}$$

or $k = 2$. Thus, the only value of k for which the two volumes are the same is $k = 2$.