AP Calculus 2012 AB FRQ Solutions

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1 Problem 1

1.1 Part a

According to the data in the table, $W(15) = 67.9^\circ$ F, while $W(9) = 61.8^\circ$ F. Therefore,

$$W'(12) \sim \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6} = \frac{61}{60}. \tag{1}$$

This means that, 12 minutes after the heating began, the temperature of the water in the tub is increasing at roughly $61/60$ degrees Fahrenheit per minute.

1.2 Part b

By the Fundamental Theorem of Calculus,

$$\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16.0. \tag{2}$$

Thus, the water temperature has increased by about 16.0° F. during the first twenty minutes of heating.

1.3 Part c

Using a left Riemann sum and the data from the table, we can approximate

$$\frac{1}{20}\int_0^{20} W(t)\,dt \sim \frac{1}{20}\left[55.0\cdot(4-0)+57.1\cdot(9-4)+61.8\cdot(15-9)+67.9\cdot(20-15)\right]. \tag{3}$$

The value of this sum is 60.79° F. We were given that W is an increasing function on the interval in question, so the value of $W(t)$ at the left-hand end-point of each of the subintervals we have used is the minimum of $W(t)$ in that subinterval. Consequently, the left Riemann sum underestimates the integral for the average value of W.

1.4 Part d

By the Fundamental Theorem of Calculus,

$$W(25) = W(20) + \int_{20}^{25} W'(t)\,dt \tag{4}$$

$$= W(20) + 0.04\int_{20}^{25}\left[\sqrt{t}\cos(0.06t)\right]\,dt. \tag{5}$$

Integrating numerically, we find that $W(25) \sim 73.04315^\circ$ F.

2 Problem 2

2.1 Part a

When $t = 2$, we have

$$\frac{dx}{dt} = \frac{\sqrt{4}}{e^2} = 2e^{-2} > 0. \tag{6}$$

Moreover $x'(t) = e^{-t}\sqrt{t+2}$ is continuous near $t = 2$, so taking the positive direction of the horizontal x-axis to be rightward, as is conventional, we see that the particle is moving to the right when $t = 2$.

If $x = x(t)$ and $y = y(t)$ give a curve which is locally the graph of y as a function F of x near $t = t_0$, then, provided all of the indicated derivatives exist and $x'(t_0) \neq 0$, then we have, by the Chain Rule, $F'[x(t_0)] \cdot x'(t_0) = y'(t_0)$, whence

$$F'[x(t_0)] = \frac{y'(t_0)}{x'(t_0)}. \tag{7}$$

The slope of the path of the particle at $t - 2$ is therefore

$$F'[x(2)] = \frac{y'(2)}{x'(2)} = \frac{e^2 \sin^2 2}{2} \sim 3.05472. \tag{8}$$

2.2 Part b

By the Fundamental Theorem of Calculus, the particle's position $\mathbf{r}(t)$ at time $t = 4$ is given by

$$\mathbf{r}(t) = \langle 1, 5\rangle + \int_2^4 \langle e^{-\tau}\sqrt{\tau + 2}, \sin^2 \tau\rangle \, d\tau. \tag{9}$$

Integrating numerically, we obtain

$$\mathbf{r}(4) = \langle 1.25295, 5.56346\rangle. \tag{10}$$

2.3 Part c

The particle's speed at $t = 4$ is

$$\sqrt{[x'(4)]^2 + [y'(400)]^2} = \sqrt{6e^{-4} + \sin^4 6} \sim 0.66177. \tag{11}$$

2.4 Part d

Distance traveled is the integral of speed. Hence the required distance is

$$\int_2^4 \sqrt{[x'(4)]^2 + [y'(4)]^2} = \int_2^4 \sqrt{e^{-2\tau}(\tau + 2) + \sin^4 \tau} \, d\tau \tag{12}$$

By numerical integration, the particle travels approximately 0.65098 units during the interval $2 \le t \le 4$.

3 Problem 3

3.1 Part a

The value $g(2)$ is the negative of the area bounded by the lines $y = 0$, $y = (x - 1)/2$, and $x = 2$. The region is a triangle of base 1, altitude $1/2$, so $g(2) = -1/4$.

The value $g(-2)$ is the sum of, on the one hand, the area of the triangular region bounded by the lines $y = 0$, $y = -3(x + 1)$, and $x = -2$, and, on the other hand, the area of a semi-circular region of radius 1. Thus $g(-2) = (3 + \pi)/2$.

3.2 Part b

We have $g(x) = \int_1^x f(t)\,dt$, so it follows from the Fundamental Theorem of Calculus that $g'(x) = f(x)$. Hence $g'(-3) = f(-3)$, and we read the latter from the graph: Thus, $g'(-3) = f(-3) = 2$.

From our conclusion above that $g'(x) = f(x)$, it follows that $g''(x) = f'(x)$ wherever the latter exists. But the graph of $y = f(x)$ is a straight line of slope 1 in the vicinity of the point $(-3, 2)$, so $g''(-3) = f'(-3) = 2$.

3.3 Part c

The line tangent to $y = g(x)$ is horizontal only where $g'(x) = f(x)$ [as found above in Part b] is 0. From the graph, we see that $f(x) = 0$ in just two places: where $x = -1$ and where $x = 1$. Thus, $x = -1$ and $x = 1$ give the only horizontal tangent lines to the curve $y = g(x)$.

As x increases through $x = 1$, $g'(x) = f(x)$ doesn't change sign. By the First Derivative Test, g has neither a relative minimum nor a relative maximum at $x = 1$.

3.4 Part d

The curve $y = g(x)$ has inflection points where the second derivative, $g''(x)$, undergoes a change of sign. We saw in Part b, above, that $g''(x) = f'(x)$, and we can read the sign of the latter from the graph. Hence, g has inflection points at $x = -2$, $x = 0$, and $x = 1$.

4 Problem 4

4.1 Part a

From the table, $f'(1) = 8$. An equation for the tangent line at the point where $x = 1$ is thus

$$y = f'(1)(x-1), \text{ or} \tag{13}$$
$$y = 15 + 8(x-1). \tag{14}$$

Using the tangent line to approximate the curve near $x = 1$, we find that

$$f(1.4) \sim 15 + 8(1.4-1) = 18.2. \tag{15}$$

4.2 Part b

The midpoint of the interval $[1.0, 1.2]$ is 1.1, while the midpoint of $[1.2, 1.4]$ is 1.3. The midpoint estimate for $\int_1^{1.4} f'(x)\,dx$ using two subdivisions of equal length is thus

$$f'(1.1)\cdot(1.2-1.0) + f'(1.3)\cdot(1.4-1.2) = 12\cdot(0.2) + 13\cdot(0.2) = 25\cdot(0.2) = 5. \tag{16}$$

4.3 Part c

By the Fundamental Theorem of Calculus,

$$f(1.4) = f(1.0) + \int_{1.0}^{1.4} f'(x)\,dx. \tag{17}$$

Euler's method with two steps of equal size is equivalent to using the left-hand rule with two subdivisions of equal size to estimate $f(1.4) = f(1) + \displaystyle\int_1^{1.4} f'(t)\,dt$. Thus, we calculate

$$f.(1.4) - f(1) \sim \int_1^{1.4} f'(t)\,dt \sim 15 + 5 = 20. \tag{18}$$

4.4 Part d

The second-degree Taylor polynomial for $f(x)$ about $x = 1$ is

$$T(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \tag{19}$$

$$= 15 + 8(x-1) + 10(x-1)^2. \tag{20}$$

Thus,

$$F(1.4) \sim T(1.4) = 15 + 8 \cdot (0.4) + 10 \cdot (0.4)^2 = 19.8. \tag{21}$$

5 Problem 5

5.1 Part a

We suppose that $B(t_1) = 40$, while $B(t_2) = 70$. Because

$$B'(t) = \frac{1}{5}[100 - B(t)], \tag{22}$$

we have

$$B'(t_1) = \frac{1}{5}[100 - B(t_1)] = 12 > 5 = \frac{1}{5}[100 - B(t_2)] = B'(t_2). \tag{23}$$

It follows that the bird is growing faster when it weighs 40 grams than when it weighs 70 grams.

5.2 Part b

From $B'(t) = \frac{1}{5}[100 - B(t)]$, we obtain

$$B''(t) = -\frac{1}{5}B'(t) = -\frac{1}{25}[100 - B(t)] \tag{24}$$

But this quantity is negative when $B(t) < 100$, and, because $B(0) = 20$, this means that the graph of B must be concave downward on some interval immediately to the right of $t = 0$. The given graph doesn't have these properties, and so can't be the graph of B.

5.3 Part c

If $B(0) = 20$, then, B being the solution of a differential equation, is a continuous function and $100 - B(t) > 0$ on some open interval, I, centered at $t = 0$.

From $B'(t) = \frac{1}{5}[100 - B(t)]$, we have for all τ in I,

$$\frac{B'(\tau)}{100 - B(\tau)} = \frac{1}{5}, \text{ whence, for any } t \text{ in } I, \tag{25}$$

$$\int_0^t \frac{B'(\tau)}{100 - B(\tau)}\, d\tau = \frac{1}{5}\int_0^t d\tau. \tag{26}$$

Integrating, and making use of the fact that $B(0) = 20 < 100$, we see that

$$-\ln[100 - B(\tau)]\Big|_0^t = \frac{1}{5}\tau\Big|_0^t, \text{ or} \tag{27}$$

$$\ln 80 - \ln[100 - B(t)] = \frac{t}{5}, \text{ which we rewrite as} \tag{28}$$

$$\ln[100 - B(t)] = \ln 80 - \frac{1}{5}t. \tag{29}$$

From this it follows that

$$100 - B(t) = 80e^{-t/5}, \text{ or} \tag{30}$$

$$B(t) = 100 - 80e^{-t/5}. \tag{31}$$

6 Problem 6

6.1 Part a

We have

$$\lim_{n\to\infty}\left[\frac{|x|^{2n+3}}{2n+5}\cdot\frac{2n+3}{|x|^{2n+1}}\right] = x^2 \lim_{n\to\infty}\frac{2n+3}{2n+5} \tag{32}$$

$$= x^2 \lim_{n\to\infty}\frac{2+3/n}{2+5/n} = x^2. \tag{33}$$

If now follows, by the Ratio Test, that the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+3}$ converges when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$ the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}$ and when $x = -1$

the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+3}$. The second of these series is the negative of the first, so either they both converge or they both diverge, and it suffices to consider the first. But $\frac{1}{2n+3}$ decreases to zero as $n \to \infty$, so, by the Alternating Series Test, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3}$ converges. Thus, the interval of convergence for $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3}$ is $[-1, 1]$.

6.2 Part b

When $x = 1/2$, the magnitude of the third term of this series—which is, by the Alternating Series Test, a bound for the error in using the sum of the first two terms of the series—is

$$\frac{1}{2^5 \cdot 7} = \frac{1}{224} < \frac{1}{200}. \tag{34}$$

It follows that

$$\left| g\left(\frac{1}{2}\right) - \frac{17}{200} \right| < \frac{1}{200}. \tag{35}$$

6.3 Part c

In the interior of its interval of convergence, a power series may be differentiated term by term to obtain a power series for the derivative of the function it represents. We are given that

$$g(x) = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+3} + \cdots , \tag{36}$$

so

$$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n \frac{2n+1}{2n+3} x^{2n} + \cdots . \tag{37}$$