AP Calculus 2014 BC FRQ Solutions

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1 Problem 1

1.1 Part a

We have $A(t) = 6.687(0.931)^t$, where t is measured in days and $A(t)$ is measured in pounds. So the average rate of change of $A(t)$ over the interval $0 \le t \le 30$ is

$$\frac{A(30) - A(0)}{30 - 0} \sim \frac{0.7829279 - 6.687}{30} \sim -0.19680 \text{ pounds per day.} \tag{1}$$

1.2 Part b

We have $A'(t) = 6.687 \cdot (0.931)^t \cdot \ln(0.931) = 0.47809376 \cdot (0.931)^t$, so $A'(15) \sim -0.16359$. Thus, after 15 days have passed, the amount of grass clippings remaining in the bin is changing at about the rate of -0.164 pounds per day.

1.3 Part c

The average amount of grass clippings in the bin over the interval $0 \le t \le 30$ is

$$\frac{1}{30} \int_0^{30} A(\tau)\, d\tau,$$

so we must solve for t in the equation

$$30A(t) = \int_0^{30} A(\tau)\, d\tau. \tag{2}$$

We solve numerically and obtain $t \sim 12.41477$. Thus, we need 12.41477 days.

1.4 Part d

The linear approximation $L(t)$ to A at $t = 30$ is

$$L(t) = A(30) + A'(30)(t - 30), \text{ or} \tag{3}$$
$$L(t) \sim 0.78293 - 0.05598(t - 30). \tag{4}$$

To find the approximate time at which there will be 0.5 pounds of grass clippings remaining in the bin, we must solve the equation $L(t) = 0.5$ for t. Doing so, we find that, according to this model, there will be 0.5 pounds of grass clippings in the bin when $t = 35.05443$ days.

2 Problem 2

2.1 Part a

To find the area of the region that is inside the graphs of the polar equation $r = 3$ and the polar equation $r = 3 - 2\sin 2\theta$, with $0 \leq \theta \leq \pi$, we must first find the points where the two curves intersect, or where $3 = 3 - 2\sin 2\theta$. The solutions to this equation in the interval $[0, \pi]$ are those of the equation $\sin 2\theta = 0$, or $\theta = 0, \pi/2, \pi$. The area A in question is therefore given by

$$A = \frac{1}{2}\int_0^{\pi/2} [3 - 2\sin 2\theta]^2 \, d\theta + \frac{3^2}{4}\pi. \tag{5}$$

Numerical integration gives $A \sim 9.70796$.

Note: Symbolic integration is feasible, if a bit time-consuming:

$$\frac{1}{2}\int_0^{\pi/2} [3 - 2\sin 2\theta]^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \left[9 - 12\sin 2\theta + 4\sin^2 2\theta\right] \, d\theta \tag{6}$$
$$= \left[\frac{9}{2}\theta + 3\cos 2\theta\right]\Bigg|_0^{\pi/2} + \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta \tag{7}$$
$$= \frac{9}{4}\pi - 6 + \left[\theta - \frac{1}{4}\sin 4\theta\right]\Bigg|_0^{\pi/2} = \frac{11}{4}\pi - 6. \tag{8}$$

2.2 Part b

If $r = 3 - 2\sin 2\theta$, then

$$x = r\cos\theta = [3 - 2\sin 2\theta]\cos\theta, \text{ so} \tag{9}$$

$$\frac{dx}{d\theta} = 4\cos\theta\cos 2\theta - \sin\theta(3 - 2\sin 2\theta) \tag{10}$$

When $\theta = \pi/6$ this is

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = 4\cos\frac{\pi}{6}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}(3 - 2\sin\frac{\pi}{3}) \tag{11}$$

$$= \frac{1}{2}\left(\sqrt{3} - 1\right) \sim 1.09808 \tag{12}$$

2.3 Part c

This question is poorly stated, because "the distance between the curves" is zero. (They cross! More than once!) This is a constant and doesn't change—no matter what the first sentence of the problem says. The problem writers probably meant "the distance from one curve to the other along the ray corresponding to the value of θ". Using this interpretation, the distance, $D(\theta$, that corresponds to a given value θ is $D(\theta) = 3 - (3 - 2\sin 2\theta) = 2\sin 2\theta$. Then $D'(\theta) = 4\cos 2\theta$ gives the rate at which the distance changes with respect to θ. When $\theta = \pi/3$, we have

$$D'(\pi/3) = 4\cos\left[2 \cdot \frac{\pi}{3}\right] = -2. \tag{13}$$

Thus, the required rate of change is -2.

2.4 Part d

When

$$r = 3 - 2\sin 2\theta, \tag{14}$$

we have

$$\frac{dr}{dt} = -4\frac{d\theta}{dt}\cos 2\theta. \tag{15}$$

Thus, under the circumstances given,

$$\frac{dr}{dt} = -4 \cdot 3 \cdot \cos\frac{\pi}{3} = -6. \tag{16}$$

3 Problem 3

We are given a function graphically; it can be written as

$$f(x) = \begin{cases} -x-3, & -5 \le x \le -3; \\ \\ \dfrac{4}{3}x+4, & -3 < x \le 0; \\ \\ 4-2x, & 0 < x \le 4. \end{cases} \tag{17}$$

We put $g(x) = \displaystyle\int_{-3}^{x} f(t)\,dt$.

3.1 Part a

The integral that gives $g(3)$ is the sum of the (signed) areas of the triangle whose vertices are $(-5,2)$, $(-3,0)$, and $(-5,0)$; the triangle whose vertices are $(-3,0)$, $(2,0)$, and $(0,4)$; and the triangle whose vertices are $(2,0)$, $(3,-2)$, and $(3,0)$. Thus, $g(3) = 2+10+(-1) = 11$.

3.2 Part b

The function f is, by the Fundamental Theorem of Calculus, the function g'. So f', which gives the slope of f, is g''. Thus, g is both increasing and concave down where f is positive and f' is negative. The intervals in question are therefore $(-5,-3)$ and $(0,2)$.

3.3 Part c

With $h(x) = \dfrac{g(x)}{5x}$, we have

$$h'(x) = \frac{5xg'(x) - 5g(x)}{25x^2}, \text{ so} \tag{18}$$

$$h'(3) = \frac{\not{5}\cdot 3 \cdot g'(3) - \not{5}\cdot g(3)}{\not{5}\cdot 3} \tag{19}$$

$$= \frac{3\cdot f(3) - g(3)}{3} \tag{20}$$

$$= \frac{3\cdot(-2) - 11}{3} = -\frac{17}{3}. \tag{21}$$

3.4 Part d

If $p(x) = f\left(x^2 - x\right)$, then, by the Chain Rule,

$$p'(x) = (2x-1)f'\left(x^2 - x\right). \tag{22}$$

Thus,

$$\begin{aligned} p'(1) &= [2\cdot(-1)-1]\cdot f'\left[(-1)^2 - (-1)\right] &(23)\\ &= (-3)\cdot f'(2) = (-3)\cdot(-2) = 6. &(24)\end{aligned}$$

4 Problem 4

4.1 Part a

The average acceleration of the train over the interval $2 \le t \le 8$ is

$$\frac{v_A(8) - v_A(2)}{8-2} = \frac{(-120) - 100}{6} = -\frac{220}{6} = -\frac{110}{3} \text{ meters/min/min.} \tag{25}$$

4.2 Part b

We are given that the velocity function, v_A, is differentiable in its domain, so it is also continuous there. Now $v_A(5) = 40$ and $v_A(8) = -120$ are given in the table, so the Intermediate Value Property of continuous functions guarantees that there is a number ξ in the interval $(5, 8)$ for which $v_A(\xi) = -100$.

Note: Continuity of the derivative is not needed here; derivatives have the Intermediate Value Property—even though they need not be continuous functions. This fact is not ordinarily known to students at the level of AP Calculus, so a student who wants to use it should state it explicitly.

4.3 Part c

Under the conditions given, if $s_A(t)$ denotes the distance of train A from Origin Station at time t, then s_A is given by

$$s_A(t) = 300 + \int_2^t v_A(\tau)\, d\tau. \tag{26}$$

The train's distance from Origin Station at time $t = 12$ is thus given by

$$s_A(12) = 300 + \int_2^{12} v_A(\tau)\, d\tau. \tag{27}$$

The trapezoidal approximation, using the three subintervals given in the table, is

$$s_A(12) = 300 + \frac{1}{2}[(100+40)\cdot 3 + (40-120)\cdot 3 + (-120-150)\cdot 4] = -150 \text{ meters}. \tag{28}$$

4.4 Part d

Let $s_B(t)$ denote the distance of train B from Origin Station at time t. The distance $S(t)$ between the two trains then satisfies the equation

$$S^2 = s_A^2 + s_B^2. \tag{29}$$

Implicit differentiation with respect to t gives

$$2S(t)S'(t) = 2s_A(t)s_A'(t) + 2s_B(t)s_B'(t) \tag{30}$$

$$= 2s_A(t)v_A(t) + 2s_B(t)v_B(t). \tag{31}$$

Substituting $t = 2$ and using what has been given in the problem, we find that $S'(2) = 160$ meters per minute.

5 Problem 5

5.1 Part a

If R is the region bounded by the curves $y = xe^{x^2}$, $y = -2x$, and $x = 1$, then the area of R is

$$\int_0^1 \left[xe^{x^2} - (-2x)\right]\, dx = \left(\frac{1}{2}e^{x^2} + x^2\right)\Big|_0^1 \tag{32}$$

$$= \left(\frac{1}{2}\cdot e^1 + 1\right) - \left(\frac{1}{2}e^0 + 0\right) = \frac{1}{2}(e+1). \tag{33}$$

5.2 Part b

The volume, V, of the solid generated by rotating R about the horizontal line $y = -2$ (which passes through the lowest vertex of R) is

$$V = \pi \int_0^1 \left[(xe^{x^2} + 2)^2 - (-2x + 2)^2 \right] dx. \tag{34}$$

The integral, which need not be evaluated, is not elementary.

5.3 Part c

The lower diagonal boundary of R is a line segment that extends from $(0, 0)$ to $(1, -2)$, and has length $\sqrt{5}$. The right-hand boundary of R is a line segment that extends from $(1, -2)$ to $(1, e)$ and has length $(e + 2)$. finally, the upper boundary is the part of the curve $y = xe^{x^2}$ that extends from $(0, 0)$ to $(1, e)$ and has length

$$\int_0^1 \sqrt{1 + (y')^2}\, dx = \int_0^1 \sqrt{1 + \left(e^{x^2} + 2x^2 e^{x^2}\right)^2}\, dx. \tag{35}$$

The required perimeter, P, is given by

$$P = \sqrt{5} + (e + 2) + \int_0^1 \sqrt{1 + \left(e^{x^2} + 2x^2 e^{x^2}\right)^2}\, dx \tag{36}$$

The integral, which need not be evaluated, is not elementary.

6 Problem 6

6.1 Part a

We apply the Ratio Text:

$$\lim_{n\to\infty} \left[\left(\frac{2^{n+1}}{n+1} |x - 1|^{n+1} \right) / \left(\frac{2^n}{n} |x - 1|^n \right) \right] = |x - 1| \lim_{n\to\infty} \frac{2n}{n+1} \tag{37}$$

$$= |x - 1| \frac{2}{1 + (1/n)} = 2|x - 1|. \tag{38}$$

This limit is less than one when $|x - 1| < 1/2$, so the radius of convergence is $1/2$.

6.2 Part b

The general term of the series expansion for $f(x)$ about $x = 1$ is

$$(-1)^{n+1}\frac{2^n}{n}(x-1)^n,\ n = 1, 2, 3\ldots,$$

which we differentiate to obtain the general term for the expansion of $f'(x)$ about $x = 1$. This latter is thus

$$(-1)^{n+1}2^n(x-1)^{n-1},\ n = 1, 2, 3\ldots.$$

The series for $f'(x)$ thus begins

$$2 - 4(x-1) + 8(x-1)^2 + \cdots .$$

6.3 Part c

If the series in Part b represents $f'(x)$, then, because that series is geometric with common ratio $-2(x-1)$, we must have

$$f'(x) = \frac{1}{1+2(x-1)} = \frac{1}{2x-1} \tag{39}$$

when $|2(x-1)| < 1$. This means that

$$f(x) = \int \frac{dx}{2x-1} = \frac{1}{2}\ln|2x-1| + C, \tag{40}$$

where C is a yet to be determined constant. But we know that $|x-1| < 1/2$ for the series to be convergent. Consequently, $0 < 2x-1 < 2$, and $|2x-1| = 2x-1$. Using the original series, we find that $f(1) = 0$, and we may write

$$0 = f(1) = \frac{1}{2}\ln(2\cdot 1 - 1) + C = C. \tag{41}$$

It now follows that

$$f(x) = \frac{1}{2}\ln(2x-1) \tag{42}$$

when $|x-1| < 1/2$ or, equivalently, $\frac{1}{2} < x < \frac{3}{2}$.