AP Calculus 2018 AB FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 22, 2018

1 Problem 1

1.1 Part a

During the time interval $0 \le t \le 300$,

$$\int_0^{300} r(t)\,dt = 44\int_0^{300}\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7\,dt = 270 \tag{1}$$

people enter the line for the escalator. (Fortunately, the problem is calculator active; finding this antiderivative doesn't bear thinking about under examination conditions.)

1.2 Part b

People exit the line at the rate of 0.7 person per second, so $300 \cdot 0.7 = 210$ people leave the line between $t = 0$ and $t = 300$. Because there were 20 people in the line at $t = 0$ and 270 people entered the line between $t = 0$ and $t = 300$, there must be $20 + 270 - 210 = 80$ people in the line at $t = 300$.

1.3 Part c

In order to solve this problem, we must assume that people exit the line at the same rate (0.7 people per second) after $t = 300$; the problem statement is ambiguous about this. Under this assumption, the line will be empty for the first time when $t = 300 + 80/0.7 = 414.286$.

1.4 Part d

People arrive at the rate $r(t)$ and leave at the rate 0.7. Because there are 20 people in the line at $t = 0$, the number $n(t)$ of people in the line at time t is given, when $0 \leq t \leq 300$, by

$$n(t) = 44 \int_0^t \left(\frac{\tau}{100}\right)^3 \left(1 - \frac{\tau}{300}\right)^7 d\tau - 0.7t. \tag{2}$$

This can be minimal only when $t = 0$, $t = 300$, or $n'(t) = 0$. The latter condition is met when

$$n'(t) = 20 + 44 \left(\frac{t}{100}\right)^3 \left(1 - \frac{t}{300}\right)^7 - 0.7 = 0, \tag{3}$$

or, solving numerically, when

$$t \sim 33.0133, \tag{4}$$

and when

$$t \sim 166.5747. \tag{5}$$

Evaluating n at these four values, we find that

$$n(0) = 20, \tag{6}$$
$$n(33.0133) = 3.8034, \tag{7}$$
$$n(166.5747) = 158.0701, \tag{8}$$
$$n(300) = 80. \tag{9}$$

The minimum value of this function thus occurs at about time $t = 33.0133$. To the nearest whole number, the minimum number of people is 4.

2 Problem 2

2.1 Part a

From

$$v(t) = \frac{10 \sin\left(0.4t^2\right)}{t^2 - t + 3}, \tag{10}$$

we find that acceleration, $a(t)$, is given by

$$a(t) = v'(t) = \frac{(8t^3 - 8t^2 + 24t)\cos\left(0.4t^2\right) - (20t - 10)\sin\left(0.4t^2\right)}{(t^2 - t + 3)^2} \tag{11}$$

so that

$$a(3) \sim -2.118195, \text{ or} - 2.118. \tag{12}$$

2.2 Part b

The position, $x(t)$, of the particle at time t is given by

$$x(t) = -5 + \int_0^t \frac{10\sin\left(0.4\tau^2\right)}{\tau^2 - \tau + 3}\, d\tau. \tag{13}$$

This is not an elementary integral, and we make no attempt to find an antiderivative; when $t = 3$, a numerical integration gives

$$x(3) \sim -1.760213, \text{ or } -1.760. \tag{14}$$

2.3 Part c

We have

$$\int_0^{3.5} v(\tau)\, d\tau \sim 2.844 \tag{15}$$

$$\int_0^{3.5} |v(\tau))|\, d\tau \sim 3.737. \tag{16}$$

The first of these integrals gives the distance between the particle's positions at $t = 0$ and at $t = 3.5$. The second integral gives the total distance that the particle travels in any direction during the same interval.

2.4 Part d

If the second particle's position is $x_2(t) = t^2 - t$, then the second particle's velocity is $v_2(t) = x_2'(t) = 2t - 1$. So we must solve the equation $v(t) = v_2(t)$, or

$$\frac{10\sin\left(0.4t^2\right)}{t^2 - t + 3} = 2t - 1. \tag{17}$$

We solve numerically, and we find that the solution is $t \sim 1.571$.

Remark: Although the solution is not $\pi/2$, this number *is* correct to the required three decimal places. Will the readers accept it?

3 Problem 3

3.1 Part a

By the Fundamental Theorem of Calculus,

$$f(x) = 3 + \int_1^x g(t)\,dt, \tag{18}$$

so $g(-5)$ is 3 added to, reading from the graph, the sum of the area of a 3×3 square, the area of a triangle of base 1 and height 3 and the negative of a triangle of base 1 and height 2. That's

$$3 + 3^2 + \frac{1}{2} \cdot 1 \cdot 3 - \frac{1}{2} \cdot 1 \cdot 2 = \frac{25}{2}. \tag{19}$$

3.2 Part b

We have

$$\int_1^6 g(t)\,dt = \int_1^3 2\,dt + 2\int_3^6 (t-4)^2\,dt \tag{20}$$

$$= 2 \cdot 2 + \frac{2}{3}(t-4)^3\Big|_3^6 \tag{21}$$

$$= 4 + \frac{2}{3}[8 - (-1)] = 10. \tag{22}$$

3.3 Parts c & d

There are thorny issues with these two questions—not because they involve difficult mathematics, but because there is no general agreement about a formal definition for the term "concave upward" (or for its sibling term "concave downward"). And so, of course, there can't be general agreement about what an "inflection point" is, either. To make matters worse, even those who agree on their definitions for the two flavors of concavity disagree about what inflections points are. (Some insist that there must be a line tangent to the original curve at a point if it is to qualify as an inflection point; others omit this requirement.)

Some people take a region of upward concavity to be a region (which may, or may not, be required to be open, depending on whom we're reading) where the derivative is increasing, others a region where the tangent line lies below the curve near the point of tangency. Some take positivity of the second derivative to be (to *be*—not to *imply*) upward concavity. Still others define a function F to be concave upward on an interval I provided that for any pair $x_1 < x_2$ of points in I and any number α between 0 and 1 it is true that $F[\alpha x_1 + (1-\alpha)x_2] \leq \alpha F(x_1) + (1-\alpha)F(x_2)$—that is, no matter what points x_1 and x_2 we choose in I, the curve $y = F(x)$ never rises above the line segment connecting the points $\big(x_1, F(x_1)\big)$ and $\big(x_2, F(x_2)\big)$.

Figures 1 and 2 show the graphs of f and $f'' = g'$ respectively. It is clear from the graph that if we adopt the last definition we gave for upward concavity—the one in terms of line segments lying above the curve—then f is concave upward on $[-5, 3]$ and on $[4, 6]$.

If, on the other hand, we think that a function is concave upward just in those regions where its second derivative is positive, we must conclude that f is concave upward on the intervals $(-2, -1)$, $(0, 1)$, and $(4, 6)$.

That folks use the term "increasing" in different ways adds a bit of spice. (Some folks require that $f(\alpha) < f(\beta)$ when $\alpha < \beta$ and both are in I, some only that $f(\alpha) \leq f(\beta)$, for f to be increasing on .

It should be clear now that someone can reasonably assert that f is increasing and concave upward on the intervals $(-1, 3)$ and $(4, 6)$ while someone else can assert—just as reasonably—that f is both increasing and concave upward just on the intervals $(0, 1)$ and $(4, 6)$. It is only when we know precisely what these people are using for their definitions that we can decide whether they're right or wrong.

Different (correct) decisions about different meanings for upward and downward concavity can clearly lead to different (correct) decisions about inflection points.

Students in elementary calculus will almost certainly have seen just one definition for each of their concepts; they won't even know that different people might use substantially different definitions. So it's very unlikely that these students will refer to their definitions—let alone state them in full—in the answers they give to these questions.

All definitions of concavity that I've seen yield the theorem that tells us that a function whose second derivative is positive on an interval is necessarily concave upward on that interval. My guess is that the development committee looked (and the readers will look) at $f''(x)$, observed (will observe) that it changes sign at just one point—at $x = 4$—and concluded (will conclude) that there is just one inflection point, which is at $x = 4$.

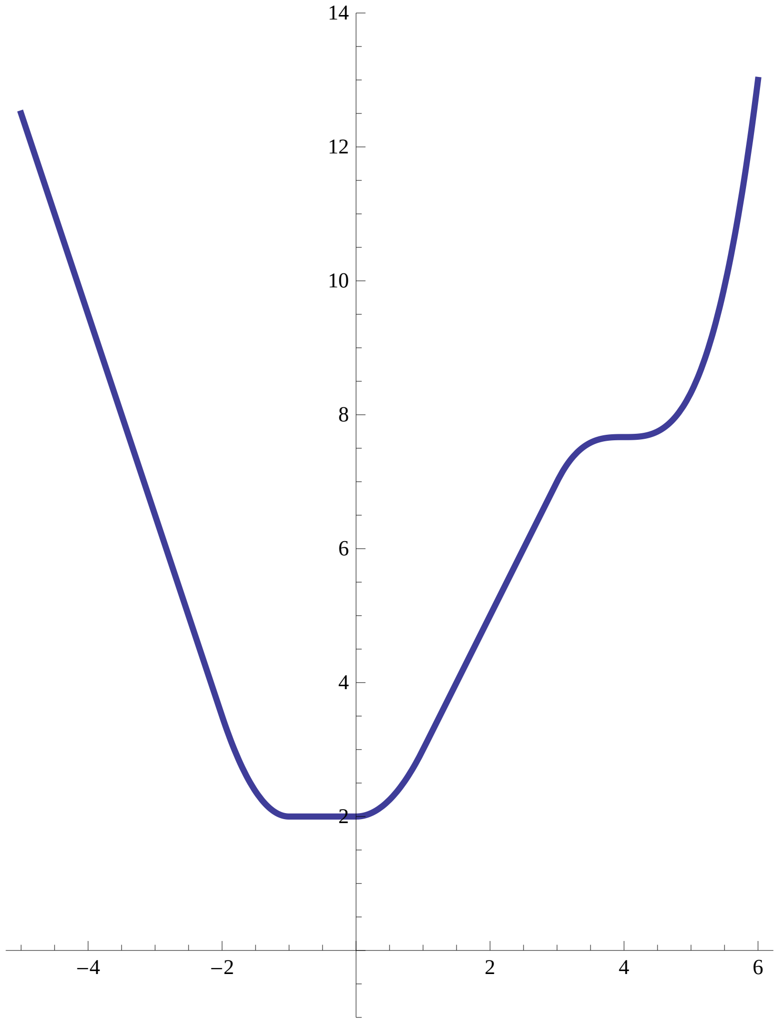

Figure 1: $y = f(x) = 3 + \int_1^x g(t)\, dt$

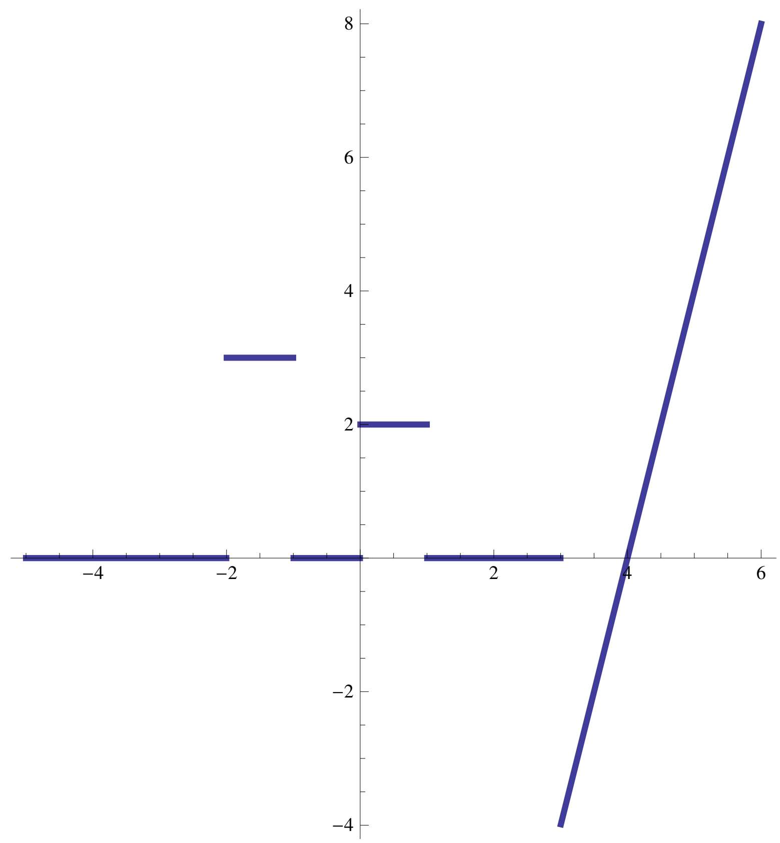

Figure 2: The graph of $f''(x) = g'(x)$

It's very unusual for the development committee to pose such an ambiguous problem. What the readers will do with the mess is anybody's guess. I'm glad I don't have to decide.

4 Problem 4

4.1 Part a

From the table, we estimate

$$H'(6) \sim \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{7 - 5} = \frac{5}{2}. \tag{23}$$

The number $H'(6)$ gives, in meters per year, the rate at which the tree is growing at time $t = 6$.

4.2 Part b

The function H is given twice differentiable, presumably (though the problem statement is vague about this) at least on the interval $(2, 10)$, so it is continuous on the interval $[3, 5]$ and differentiable on the interval $(3, 5)$. The hypotheses of the mean value theorem being satisfied, there must be at least one point ξ in the interval $(3, 5)$, and therefore in the interval $(2, 10)$, such that

$$H'(\xi) = \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{5 - 3} = \frac{4}{2} = 2. \tag{24}$$

4.3 Part c

The required trapezoidal sum is $\dfrac{1}{10-2} = \dfrac{1}{8}$ times

$$\begin{aligned} &\frac{1}{2}[H(2) + H(3)](3 - 2) + \frac{1}{2}[H(3) + H(5)](5 - 3) \\ &\qquad + \frac{1}{2}[H(5) + H(7)](5 - 7) + \frac{1}{2}[H(10) + H(7)](10 - 7) \\ &= \frac{1}{2}[(1.5 + 2)(3 - 2) + (2 + 6)(5 - 3) + (6 + 11)(7 - 5) + (11 + 15)(10 - 7)] \qquad (25) \\ &= \frac{1}{2}(3.5 \cdot 1 + 8 \cdot 2 + 17 \cdot 2 + 26 \cdot 3) = 65.75. \qquad (26) \end{aligned}$$

The average height during the period $2 \le t \le 10$ is therefore 8.219 meters, to three decimal places.

4.4 Part d

We are given

$$G(x) = \frac{100x}{1+x}, \tag{27}$$

where x is the diameter of the base of the tree in meters and $h = G(x)$ is the height, also in meters, of the tree. We have

$$\frac{dh}{dt} = \frac{dh}{dx} \cdot \frac{dx}{dt}. \tag{28}$$

Now

$$\frac{dh}{dx} = \frac{d}{dx}\left(\frac{100x}{1+x}\right) = \frac{100}{(1+x)^2}, \tag{29}$$

and $h = 50$ implies that $x = 1$, so that we have been given $\left.\frac{dx}{dt}\right|_{h=50} = 0.03$. Putting this all together, we find that

$$\left.\frac{dh}{dt}\right|_{h=50} = \frac{100}{(1+1)^2} \cdot 0.03 = 0.75 \text{ meters/year}. \tag{30}$$

5 Problem 5

Let f be the function given by $f(x) = e^x \cos x$.

5.1 Part a

The average rate of change of f on the interval $0 \leq x \leq \pi$ is

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{e^\pi \cos \pi - e^0 \cos 0}{\pi} = \frac{-e^\pi - 1}{\pi}. \tag{31}$$

5.2 Part b

$f'(x) = e^x(\cos x - \sin x)$, so the slope of the line tangent to the curve $y = f(x)$ at the point corresponding to $x = 3\pi/2$ is

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2}\left(\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}\right) = e^{3\pi/2}. \tag{32}$$

5.3 Part c

The absolute minimum value of $f(x)$ on $[0, 2\pi]$ must occur at a critical point or at an endpoint of the interval. The function f is everywhere differentiable, so the critical points are where the derivative vanishes—which, from the value of $f'(x)$ we found in Part b of this problem, can happen only where $\cos x = \sin x$, or, because $\sin x$ never vanishes where $\cos x$ does, where $\tan x = 1$. Thus, the critical points in $[0, 2\pi]$ are at $x = \pi/4$ and $x = 5\pi/4$.

We find the absolute minimum by examining $f(0)$, $f(\pi/4)$, $f(5\pi/4)$, and $f(2\pi)$. Now $e^x > 0$ for all x; and $\cos x$ is positive at all of these values but $5\pi/4$, where it is negative. It follows that the absolute minimum value of $f(x) = e^x \cos x$ on the interval $[0, 2\pi]$ is found at $x = 5\pi/4$ and is $f(5\pi/4) = -e^{5\pi/4}/\sqrt{2}$.

5.4 Part d

We have $\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} [e^x \cos x] = 0$. We are given that g is differentiable, and it is therefore continuous at $x = \pi/2$. So $\lim_{x\to\pi/2} g(x) = g(\pi/2) = 0$. Hence, by l'Hôpital's rule,

$$\lim_{x\to\pi/2} \frac{f(x)}{g(x)} = \lim_{x\to\pi/2} \frac{f'(x)}{g'(x)}, \tag{33}$$

provided that the latter limit exists. From the graph given with the problem, we see that $\lim_{x\to\pi/2} g'(x) = 2$. On the other hand,

$$\lim_{x\to\pi/2} f'(x) = \lim_{x\to\pi/2} [e^x(\cos x - \sin x)] = -e^{\pi/2}. \tag{34}$$

Therefore,

$$\lim_{x\to\pi/2} \frac{f(x)}{g(x)} = \lim_{x\to\pi/2} \frac{f'(x)}{g'(x)} = -\frac{e^{\pi/2}}{2}. \tag{35}$$

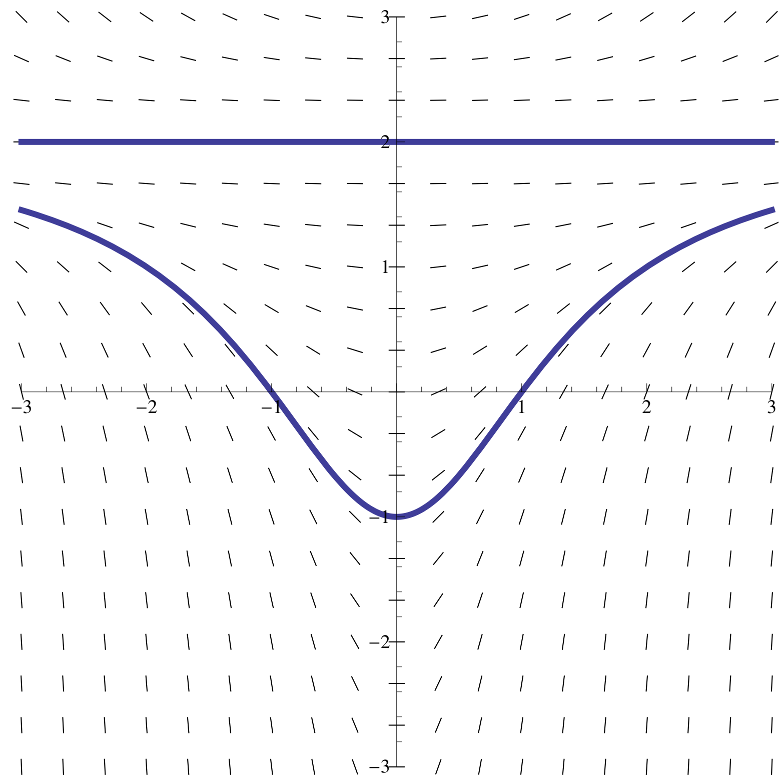

Figure 3: Two solutions of $y' = x(y-1)^2/3$.

6 Problem 6

6.1 Part a

6.2 Part b

If $y = f(x)$ is the particular solution to the differential equation that satisfies $f(1) = 0$, then

$$f'(1) = \frac{1}{3} \cdot 1 \cdot [f(1) - 2]^2 = \frac{4}{3}. \tag{36}$$

An equation for the line tangent to the solution at $x = 1$ is therefore

$$y = f(1) + f''(1)(x - 1), \tag{37}$$

or

$$y = \frac{4}{3}(x - 1). \tag{38}$$

The value of y on this tangent line approximates $f(0.7)$. On the tangent line, we have

$$y\Big|_{x=0.7} = \frac{4}{3}(0.7 - 1) = -0.4, \tag{39}$$

so we conclude that $f(0.7) \sim -0.4$.

6.3 Part c

Supposing, as in the previous part of this problem, that $y = f(x)$ is the solution we seek, we have

$$f'(x) = \frac{1}{3}x[f(x) - 2]^2, \tag{40}$$

or

$$\frac{f'(x)}{[f(x) - 2]^2} = \frac{x}{3}. \tag{41}$$

Being about to use the variable x in another role, we rewrite this equation in terms of the new variable, t:

$$\frac{f'(t)}{[f(t) - 2]^2} = \frac{t}{3}. \tag{42}$$

We integrate both sides of this latter equation from 1 to x and solve for $f(x)$:

$$\int_1^x \frac{f'(t)}{[f(t)-2]^2}\,dt = \int_1^x \frac{t}{3}\,dt; \tag{43}$$

$$\left.\frac{1}{2-f(t)}\right|_1^x = \left.\frac{t^2}{6}\right|_1^x; \tag{44}$$

$$\frac{1}{2-f(x)} - \frac{1}{2-f(1)} = \frac{x^2-1}{6}; \tag{45}$$

$$\frac{1}{2-f(x)} = \frac{x^2+2}{6}; \tag{46}$$

$$f(x) = \frac{2(x^2-1)}{x^2+2}. \tag{47}$$

Note: Substitute $u = f(t)$, $du = f'(t)\,dt$ on the left side of equation (43). If $t = 1$, then $u = f(1) = 0$, while if $t = x$, then $u = f(x) = y$. Equation (43) then becomes

$$\int_0^y \frac{du}{(u-2)^2} = \int_1^x \frac{t}{3}\,dt, \tag{48}$$

which is what we see in the (less rigorous but perhaps more familiar) separation of variables technique.