Probability and Statistics

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January 26, 2019

Notes

Topics

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Objectives

Objectives:

- Identify the sample space and events associated with a random experiment
- Construct Venn diagrams
- Know the three probability axioms
- Know and be able to use various probability rules

Notes

Introduction to Probability (2.1, 2.2)

- A ***random experiment*** is any action or process whose outcome is uncertain.
- The ***sample space*** of a random experiment, denoted $\mathcal{S}$, is its set of **possible outcomes**.

Example

- Toss a coin. Then the sample space consists of two outcomes,

$$\mathcal{S} = \{H, T\}.$$

- Roll a six-sided die. Then the sample space consists of the six outcomes

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

Notes

Example

Randomly select a person from a population. Then the sample space consists of the individuals in the population:

$$\mathcal{S} = \{\text{Stephanie Lawson, Jeffrey Miller, Angela DuPont}, \ldots, \text{Karl Stevenson}\}$$

Notes

Example

Computer microchips are tested for correct performance at a manufacturing plant before being plugged into circuit boards.

If three randomly selected chips are tested, the sample space is

$$\mathcal{S} = \{GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB\}$$

where G represents good and B represents bad.

Notes

- An ***event*** is a collection of outcomes (subset) of the sample space. Events are denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, etc.

Example (Cont'd)

Some events associated with testing three microchips are:

$$A = \{GGG, GGB, GBG, BGG\} = \text{The event that two or more are good}$$

$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

Notes

- The ***complement*** of an event A, denoted $\boldsymbol{A'}$, is the set of outcomes in $\mathcal{S}$ that are **not** in A.
- The ***union*** of two events A and B, denoted $\boldsymbol{A \cup B}$, is the set of outcomes that are in A **or** in B.
- The ***intersection*** of two events A and B, denoted $\boldsymbol{A \cap B}$, is the set of outcomes that are in A **and** in B.

Notes

Example (Cont'd)

Consider again the events

$$A = \{GGG, GGB, GBG, BGG\} = \text{The event that two or more are good}$$

$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

Then the **complement** of A is

$$A' = \{GBB, BGB, BBG, BBB\}$$

The **union** of A and B is

$$A \cup B = \{GGG, GGB, GBG, BGG, BBB\}$$

Notes

and the **intersection** of A and B is

$$A \cap B = \{GGG\}$$

Notes

- Two events A and B are ***mutually exclusive*** (or ***disjoint***) if they have no outcomes in common, i.e. if

$$A \cap B = \emptyset$$

where $\emptyset$ is the ***null event*** containing no outcomes.

Example (Cont'd)

The events

$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

$$C = \{GBB, BGB, BBG\} = \text{The event that exactly one is good}$$

are mutually exclusive.

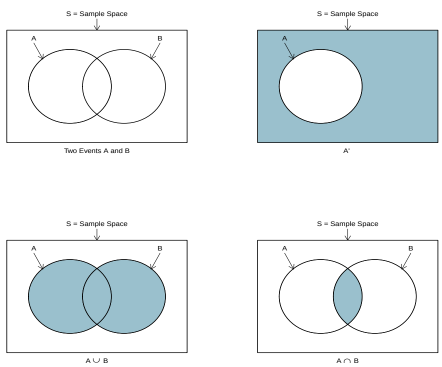
Notes

- The union and intersection operators can be applied to **more than two events**. For three events A, B, and C:
  - $\boldsymbol{A \cup B \cup C}$ is the set of outcomes that are in A **or** in B **or** in C.
  - $\boldsymbol{A \cap B \cap C}$ is the set of outcomes that are in A **and** in B **and** in C.
- The definition of *mutually exclusive* can be extended to more than two events. Events $A_1, A_2, \ldots, A_n$ are **mutually exclusive** (or ***disjoint***) if no two of them have any outcomes in common.

Notes

- A ***Venn diagram*** shows the sample space as a rectangle and events as circular regions within the rectangle.

Notes



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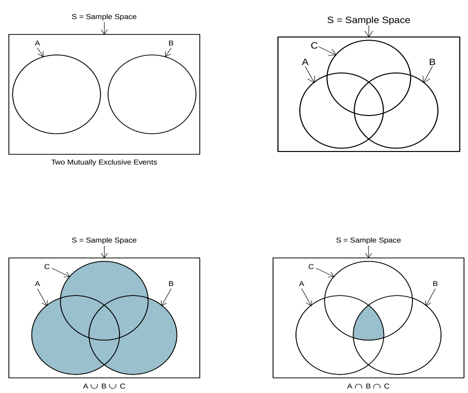


Notes

- The ***probability*** of an event A, denoted $\boldsymbol{P(A)}$, is the long-run proportion of times that A occurs.

Interpretation of Probability:

$$P(A) \;=\; \lim_{n\to\infty} \frac{n_A}{n},$$

where $\boldsymbol{n_A}$ is the number of times A occurs in the first n trials of the experiment.

Notes

- The following form the *starting point for all of probability theory*.

Probability Axioms:

1. For any event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. If $A_1, A_2, A_3 \ldots$ is an infinite sequence of **mutually exclusive** events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

Notes

- The following proposition can be proved using just the probability axioms.

Proposition

If $\emptyset$ is the *null event* containing no outcomes, then

$$P(\emptyset) = 0.$$

Notes

- The third axiom applies to a **finite** number k of (**mutually exclusive**) events (by declaring $A_{k+1}, A_{k+2}, \ldots$ to be $\emptyset$).

  For example, for two mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B)$$

  and for three mutually exclusive events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

Notes

Example

Suppose that **33%** of the people have type O$^+$ blood and **7%** have O$^-$.

Then for a randomly selected person, letting

$$A = \text{The person has type O}^+ \text{ blood}$$
$$B = \text{The person has type O}^- \text{ blood}$$

the probability that the person will have type O blood is

$$P(A \cup B) = P(A) + P(B) = 0.33 + 0.07 = 0.40.$$

Notes

- These next propositions can be proved using the three axioms.

Proposition

For any event A,

$$P(A) + P(A') \;=\; 1.$$

so

$$P(A') \;=\; 1 - P(A).$$

Proposition

For any event A,

$$P(A) \;\leq\; 1$$

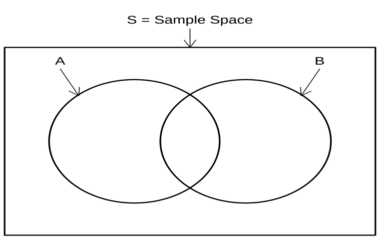
Notes

Proposition

For any two events A and B,

$$P(A \cup B) \;=\; P(A) + P(B) - P(A \cap B).$$

Notes



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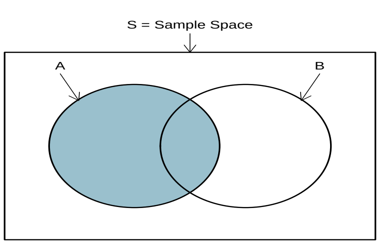

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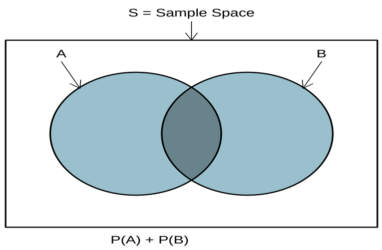

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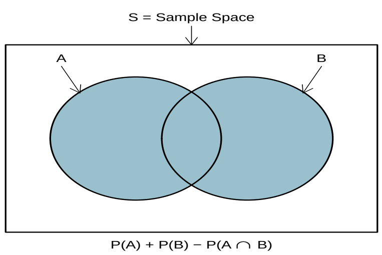

Notes

Example

In a group of students, **25%** smoke cigarettes, **60%** drink alcohol, and **15%** do both. Then for a randomly selected student, letting

$A =$ The student smokes $\quad B =$ The student drinks

the probability that the student has at least one of these bad habits is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.25 + 0.60 - 0.15 \\ &= 0.70. \end{aligned}$$

Notes

Proposition

For any three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C). \end{aligned}$$

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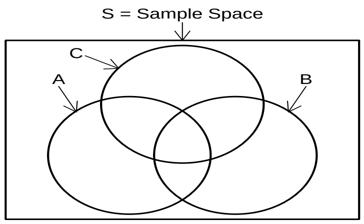

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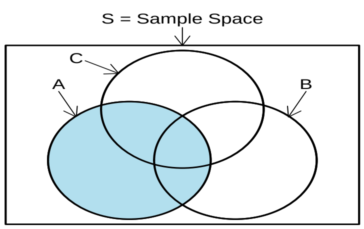

P(A)

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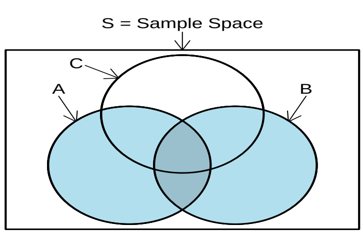
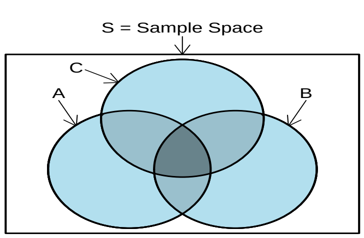


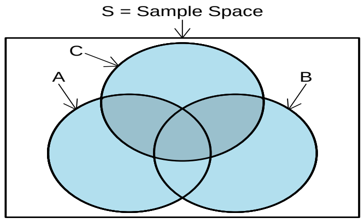
P(A) + P(B)

Notes



P(A) + P(B) + P(C)

Notes



P(A) + P(B) + P(C)
− P(A ∩ B)

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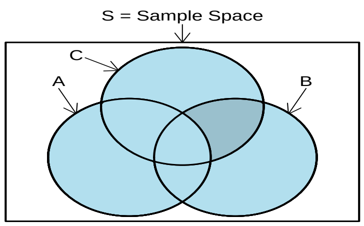
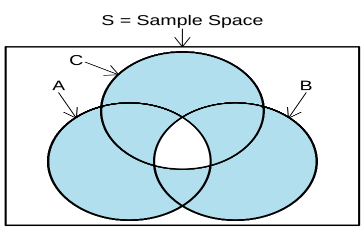


P(A) + P(B) + P(C)
− P(A ∩ B) − P(A ∩ C)

Notes



P(A) + P(B) + P(C)
− P(A ∩ B) − P(A ∩ C) − P(B ∩ C)

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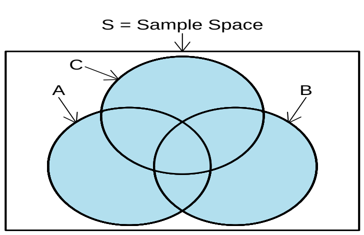

P(A) + P(B) + P(C)
− P(A ∩ B) − P(A ∩ C) − P(B ∩ C)
+ P(A ∩ B ∩ C)

Notes