# Probability and Statistics

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January 26, 2019

# Topics

# Objectives

Objectives:

- Identify the sample space and events associated with a random experiment
- Construct Venn diagrams
- Know the three probability axioms
- Know and be able to use various probability rules

# Introduction to Probability (2.1, 2.2)

- A ***random experiment*** is any action or process whose outcome is uncertain.
- The ***sample space*** of a random experiment, denoted $\mathcal{S}$, is its set of **possible outcomes.**

## Example

- Toss a coin. Then the sample space consists of two outcomes,
$$\mathcal{S} = \{H, T\}.$$
- Roll a six-sided die. Then the sample space consists of the six outcomes
$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

## Example

Randomly select a person from a population. Then the sample space consists of the individuals in the population:

$$\begin{aligned} \mathcal{S} &= \{\text{Stephanie Lawson}, \\ &\quad \text{Jeffrey Miller}, \\ &\quad \text{Angela DuPont}, \\ &\quad \vdots \\ &\quad \text{Karl Stevenson}\} \end{aligned}$$

## Example

Computer microchips are tested for correct performance at a manufacturing plant before being plugged into circuit boards.

If three randomly selected chips are tested, the sample space is

$$\mathcal{S} = \{GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB\}$$

where $G$ represents good and $B$ represents bad.

- An ***event*** is a collection of outcomes (subset) of the sample space. Events are denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, etc.

### Example (Cont'd)

Some events associated with testing three microchips are:

$$A = \{GGG, GGB, GBG, BGG\} = \text{The event that two or more are good}$$

$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

- The ***complement*** of an event $A$, denoted $\boldsymbol{A'}$, is the set of outcomes in $S$ that are **not** in $A$.
- The ***union*** of two events $A$ and $B$, denoted $\boldsymbol{A \cup B}$, is the set of outcomes that are in $A$ **or** in $B$.
- The ***intersection*** of two events $A$ and $B$, denoted $\boldsymbol{A \cap B}$, is the set of outcomes that are in $A$ **and** in $B$.

## Example (Cont'd)

Consider again the events

$$A = \{GGG, GGB, GBG, BGG\} = \text{The event that two or more are good}$$

$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

Then the **complement** of $A$ is

$$A' = \{GBB, BGB, BBG, BBB\}$$

The **union** of $A$ and $B$ is

$$A \cup B = \{GGG, GGB, GBG, BGG, BBB\}$$

and the **intersection** of $A$ and $B$ is

$$A \cap B \;=\; \{GGG\}$$

- Two events $A$ and $B$ are ***mutually exclusive*** (or ***disjoint***) if they have no outcomes in common, i.e. if

$$A \cap B = \emptyset$$

where $\emptyset$ is the ***null event*** containing no outcomes.

### Example (Cont'd)

The events

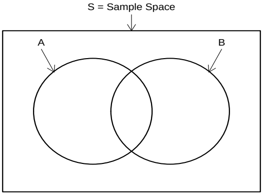
$$B = \{GGG, BBB\} = \text{The event that all three are the same}$$

$$C = \{GBB, BGB, BBG\} = \text{The event that exactly one is good}$$

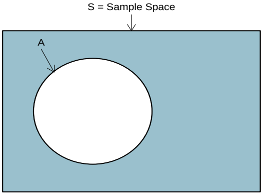
are mutually exclusive.

- The union and intersection operators can be applied to **more than two events.** For three events $A$, $B$, and $C$:
  - $\boldsymbol{A \cup B \cup C}$ is the set of outcomes that are in $A$ **or** in $B$ **or** in $C$.
  - $\boldsymbol{A \cap B \cap C}$ is the set of outcomes that are in $A$ **and** in $B$ **and** in $C$.
- The definition of *mutually exclusive* can be extended to more than two events. Events $A_1, A_2, \ldots, A_n$ are **mutually exclusive** (or ***disjoint***) if no two of them have any outcomes in common.

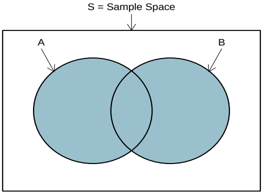
- A ***Venn diagram*** shows the sample space as a rectangle and events as circular regions within the rectangle.



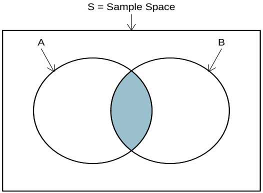
Two Events A and B



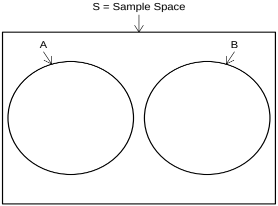
A'



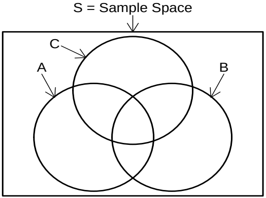
$A \cup B$

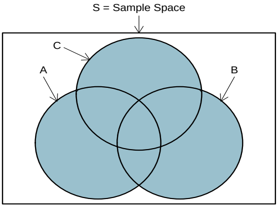


$A \cap B$

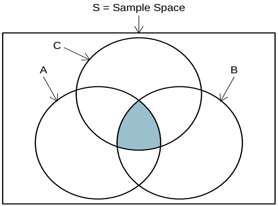



Two Mutually Exclusive Events




$A \cup B \cup C$




$A \cap B \cap C$

- The ***probability*** of an event $A$, denoted $\boldsymbol{P(A)}$, is the long-run proportion of times that $A$ occurs.

**Interpretation of Probability**:

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n},$$

where $\boldsymbol{n_A}$ is the number of times $A$ occurs in the first $n$ trials of the experiment.

- The following form the *starting point for all of probability theory*.

**Probability Axioms**:

1. For any event $A$, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. If $A_1, A_2, A_3 \ldots$ is an infinite sequence of **mutually exclusive** events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

- The following proposition can be proved using just the probability axioms.

**Proposition**

If $\emptyset$ is the *null event* containing no outcomes, then

$$P(\emptyset) \;=\; 0.$$

- The third axiom applies to a **finite** number $k$ of (**mutually exclusive**) events (by declaring $A_{k+1}, A_{k+2}, \ldots$ to be $\emptyset$).

  For example, for two mutually exclusive events $A$ and $B$,

  $$P(A \cup B) \;=\; P(A) + P(B)$$

  and for three mutually exclusive events $A$, $B$, and $C$,

  $$P(A \cup B \cup C) \;=\; P(A) + P(B) + P(C).$$

## Example

Suppose that **33%** of the people have type $O^+$ blood and **7%** have $O^-$.

Then for a randomly selected person, letting

$$\begin{aligned} A &= \text{The person has type } O^+ \text{ blood} \\ B &= \text{The person has type } O^- \text{ blood} \end{aligned}$$

the probability that the person will have type O blood is

$$P(A \cup B) = P(A) + P(B) = 0.33 + 0.07 = 0.40.$$

- These next propositions can be proved using the three axioms.

**Proposition**

For any event $A$,

$$P(A) + P(A') = 1.$$

so

$$P(A') = 1 - P(A).$$

**Proposition**

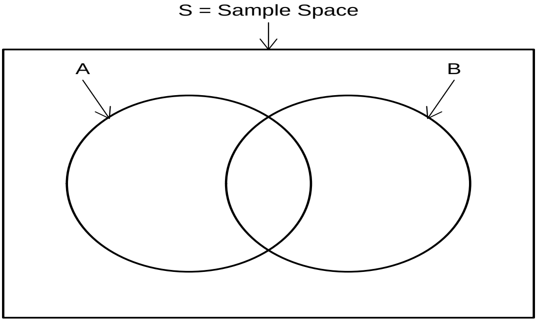
For any event $A$,

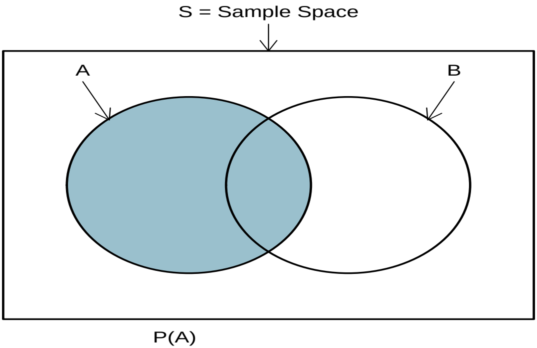
$$P(A) \leq 1$$

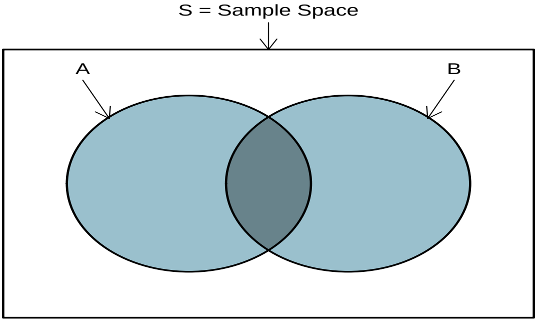
### Proposition

For any two events $A$ and $B$,

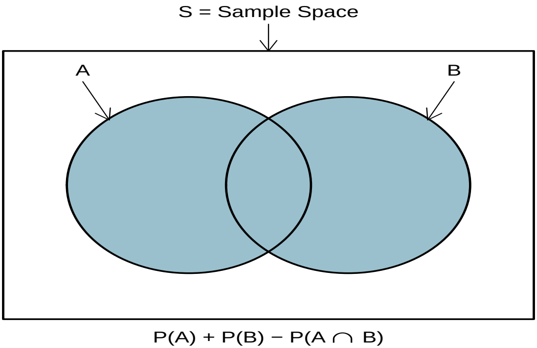
$$P(A \cup B) \ = \ P(A) + P(B) - P(A \cap B).$$


S = Sample Space
A
B


S = Sample Space
A
B
P(A)




P(A) + P(B)


S = Sample Space
A
B
P(A) + P(B) − P(A ∩ B)

## Example

In a group of students, **25%** smoke cigarettes, **60%** drink alcohol, and **15%** do both. Then for a randomly selected student, letting

$$A = \text{The student smokes} \qquad B = \text{The student drinks}$$

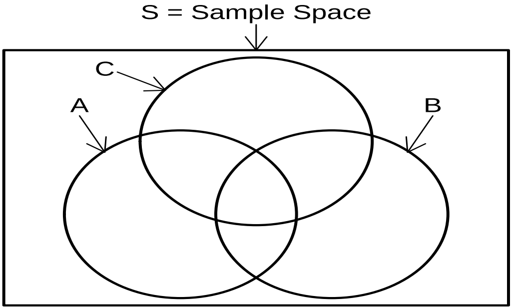
the probability that the student has at least one of these bad habits is

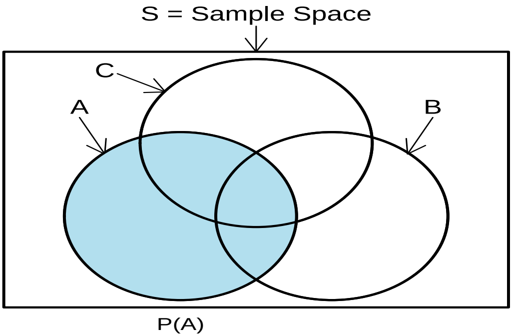
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.25 + 0.60 - 0.15 \\ &= 0.70. \end{aligned}$$

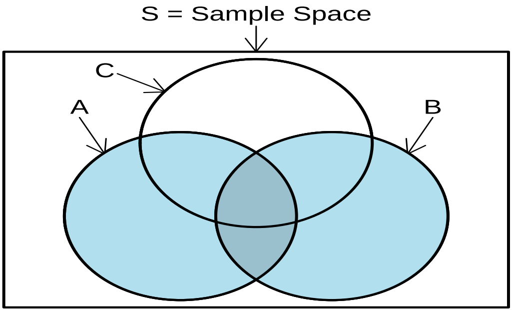
**Proposition**

For any three events $A$, $B$, and $C$,

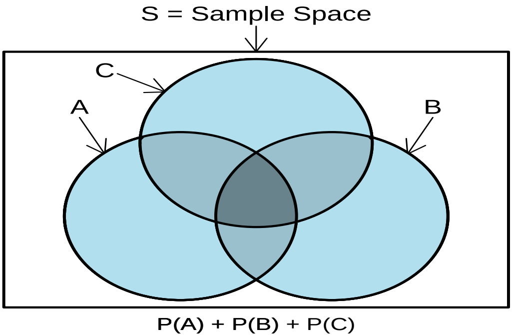
$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C). \end{aligned}$$

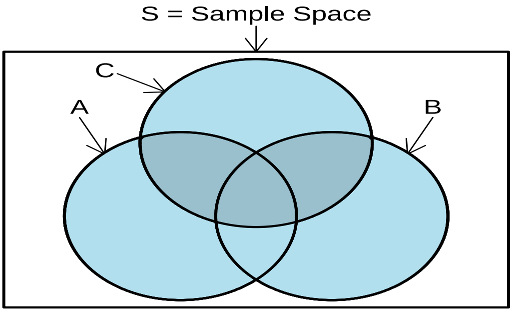

S = Sample Space
C
A
B


S = Sample Space
C
A
B
P(A)

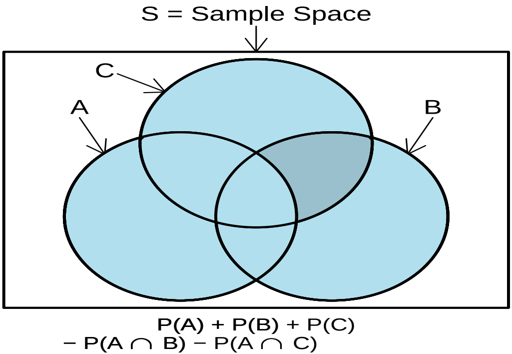


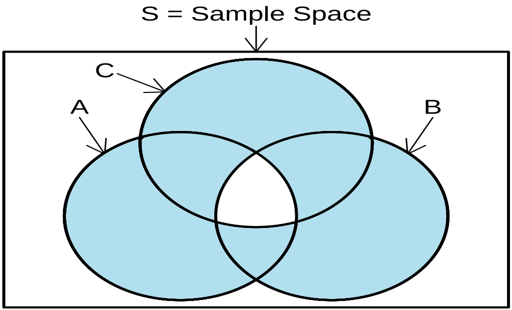

P(A) + P(B)


S = Sample Space
C
A
B
P(A) + P(B) + P(C)

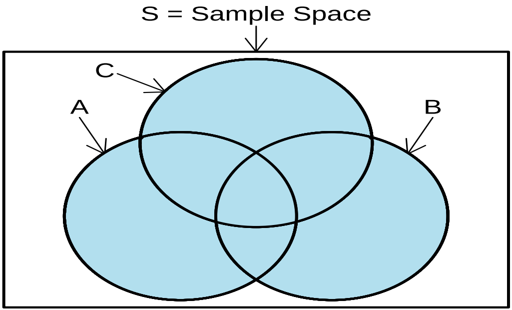



$$P(A) + P(B) + P(C) - P(A \cap B)$$


S = Sample Space
C
A
B
P(A) + P(B) + P(C)
− P(A ∩ B) − P(A ∩ C)




$P(A) + P(B) + P(C)$
$- P(A \cap B) - P(A \cap C) - P(B \cap C)$




$$P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$