Probability and Statistics

Nels Grevstad

Metropolitan State University of Denver

ngrevsta@msudenver.edu

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Topics

Objectives

Objectives:

- Count permutations and combinations
- Compute and interpret conditional probabilities
- Recognize independence and use it to compute probabilities
- Know and be able to use various probability rules

Counting Techniques (2.3)

- When the outcomes of a random experiment are equally likely,

$$P(A) = \frac{N(A)}{N}$$

where where $\boldsymbol{N(A)}$ is the number of outcomes in A and $\boldsymbol{N}$ total number of outcomes in $\mathcal{S}$.

Counting Techniques (2.3)

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  where where $N(A)$ is the number of outcomes in A and N total number of outcomes in $\mathcal{S}$.

- So techniques for counting outcomes are sometimes useful for computing probabilities.

The Product Rule: Suppose that k actions are performed in sequence and that the first action has n_1 possible results, the second has n_2 possible results, etc. Then

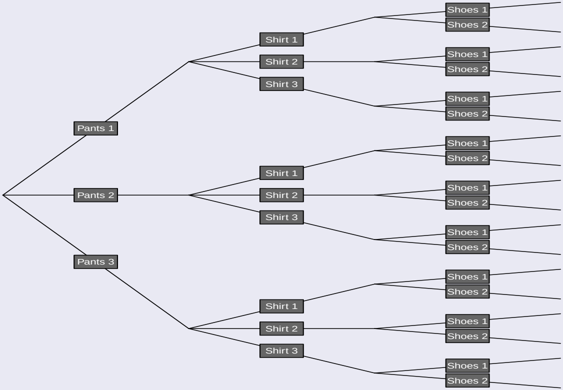
$$\text{Number of Possible Sequences of Results} = n_1 \, n_2 \, n_3 \cdots n_k$$

Example

A man has 3 pairs of pants, 3 shirts, and 2 pairs of shoes. He can get dressed in

$$n_1 n_2 n_3 \;=\; (3)(3)(2) \;=\; 18$$

ways.


Pants 1
Pants 2
Pants 3
Shirt 1
Shirt 2
Shirt 3
Shirt 1
Shirt 2
Shirt 3
Shirt 1
Shirt 2
Shirt 3
Shoes 1
Shoes 2
Shoes 1
Shoes 2
Shoes 1
Shoes 2
Shoes 1
Shoes 2
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Shoes 2

- A ***permutation*** of n objects taken k at a time is a particular **ordered** group of k of those n objects.

Example

Ten runners start a race:

Don Ben Ron Lee Bob Joe Tim Lou Ray Gil

Each assignment of 1st, 2nd, and 3rd place medals is a **permutation** of the 10 runners taken 3 at a time. Here are **three permutations**:

1st	**2nd**	**3rd**
Ron	Ray	Joe

1st	**2nd**	**3rd**
Lou	Tim	Ben

1st	**2nd**	**3rd**
Ray	Joe	Ron

Permutations: The number of different permutations of n objects taken k at a time is:

$$\begin{aligned}\text{Number of Permutations} &= n(n-1)(n-2)\cdots(n-k+1) \\ &= \frac{n!}{(n-k)!}\end{aligned}$$

where $\boldsymbol{n!}$, or $\boldsymbol{n}$ ***factorial***, is defined as

$$n! = n(n-1)(n-2)\cdots 1$$

and

$$0! = 1.$$

Example (Cont'd)

Ten runners start a race:

Don Ben Ron Lee Bob Joe Tim Lou Ray Gil

Medals for 1st, 2nd, and 3rd place can be awarded in

$$\frac{n!}{(n-k)!} = \frac{10!}{(10-3)!} = (10)(9)(8) = 720$$

ways.

- The number of permutations of n **objects taken n at a time** is just the number of orderings of the n objects and is given by

$$\frac{n!}{(n-n)!} \;=\; n!$$

(Recall that $0!$ is defined to be 1.)

Example

Five people can stand in line in

$$n! \;=\; 5! \;=\; (5)(4)(3)(2)(1) \;=\; 120$$

ways.

- A ***combination*** of n objects taken k at a time is a particular **unordered** group of k of those n objects.

Example

Ten runners are trying out for a 3-person cross-country team:

Don Ben Ron Lee Bob Joe Tim Lou Ray Gil

Each choice of 3 of the 10 runners is a **combination.** Here are **two combinations**:

Team		
Ron	Ray	Joe

Team		
Lou	Tim	Ben

~~**Team**~~		
~~Ray~~	~~Joe~~	~~Ron~~

Combinations: The number of different combinations of n objects taken k at a time is:

$$\text{Number of Combinations} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Combinations: The number of different combinations of n objects taken k at a time is:

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- Some intuition:
  - The number of *permutations*, $n(n-1)\cdots(n-k+1)$, counts each group $k!$ times.
  - So we need to divide it by $k!$ to get the number of *combinations*.

- The number of combinations is sometimes denoted $\boldsymbol{\begin{pmatrix} n \\ k \end{pmatrix}}$, i.e.

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

and read as "$\boldsymbol{n}$ **choose** $\boldsymbol{k}$".

Example (Cont'd)

The number of different 3-member teams that can be chosen from the 10 runners is

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{(10)(9)(8)}{(3)(2)(1)} = 120.$$

If the 3 team members are selected *randomly*, then the **probability** that Ron, Ray, and Joe are selected is

$$\begin{aligned} P(\text{Ron, Ray, Joe}) &= \frac{1}{\binom{10}{3}} \\ &= \frac{1}{120}. \end{aligned}$$

Some Properties of Combinations:

1. $\binom{n}{n} = 1$ and $\binom{n}{0} = 1$.
2. $\binom{n}{1} = n$ and $\binom{n}{n-1} = n$.
3. $\binom{n}{k} = \binom{n}{n-k}$.

Conditional Probability (2.4)

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- Sometimes the occurrence of an event B affects how likely it is that another event A will occur.
- The ***conditional probability*** of A, *given* the occurrence of B, is denoted $\boldsymbol{P(A|B)}$ and defined as:

Conditional Probability:

$$P(A|B) \;=\; \frac{P(A \cap B)}{P(B)}.$$

Example

At a large university, each of 38,847 students was cross-classified according to **gender** and **student level**:

	Undergraduate	Professional	Graduate	Total
Male	18,208	249	4,436	22,893
Female	4,436	651	2,660	15,954
				38,847

Consider randomly selecting one of the 38,847 students. Let

$A =$ The student is a graduate $\qquad B =$ The student is female

Then

$$P(B) \;=\; ? \qquad \text{and} \qquad P(A \cap B) \;=\; ?$$

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$A =$ The student is a graduate $\qquad B =$ The student is female

Then

$$P(B) = \frac{15,954}{38,847} \quad \text{and} \quad P(A \cap B) = \frac{2,660}{38,847}$$

and so the **conditional probability** that the student is a graduate student, *given* that she's female, is

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{2,660/38,847}{15,954/38,847} \end{aligned}$$

and so the **conditional probability** that the student is a graduate student, *given* that she's female, is

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{2,660/38,847}{15,954/38,847} = \frac{2,660}{15,954}
\end{aligned}
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and so the **conditional probability** that the student is a graduate student, *given* that she's female, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2,660/38,847}{15,954/38,847} = \frac{2,660}{15,954} = 0.167.$$

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Example

At a technical college all students take calculus and physics. **32%** get A's in calculus and **20%** get A's in both calculus **and** physics. For a randomly selected student, let

$$A = \text{Got an A in physics} \qquad B = \text{Got an A in calculus}$$

Then

$$P(B) = 0.32 \text{ and } P(A \cap B) = 0.20,$$

so the **conditional probability** that they got an A in physics, given that they got an A in calculus, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.20}{0.32} = 0.625.$$

- Here's some intuition behind the conditional probability formula

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$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$
    - The occurrence of B reduces the set of possible outcomes to just the ones in B.
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    - On this new sample space, the outcome is in A only if it's in $A \cap B$.

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$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

  - The occurrence of B reduces the set of possible outcomes to just the ones in B.
  - So $P(A|B)$ is a probability on a new, reduced sample space, B.
  - On this new sample space, the outcome is in A only if it's in $A \cap B$.
  - Dividing by $P(B)$ ensures that $P(B|B)$ for this new sample space equals 1.

- The definition of conditional probability yields the following rule.

Multiplication Rule for $P(A \cap B)$: For any two events A and B,

$$P(A \cap B) \;=\; P(B)P(A|B).$$

Example

According to a study of male high school athletes,

> 5% go on to play at the college level.
>
> 1.7% of those who play at the college level go on to play professionally.

For a randomly selected high school athlete, let

$$
\begin{aligned}
A &= \text{Plays professionally} \\
B &= \text{Plays in college}
\end{aligned}
$$

Then

$$P(B) \;=\; 0.05 \;\; \text{and} \;\; P(A|B) = 0.017$$

so the probability that an athlete will play in college **and** turn pro is

$$P(A \cap B) \;=\; P(B)P(A|B) \;=\; (0.05)(0.017) \;=\; 0.00085.$$

- The multiplication rule can be extended to more than two events.

Multiplication Rule (for Three Events): If A, B, and C are three events, then

$$P(A \cap B \cap C) = P(C)P(B|C)P(A|B \cap C)$$

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Multiplication Rule (for Three Events): If A, B, and C are three events, then

$$P(A \cap B \cap C) = P(C)P(B|C)P(A|B \cap C)$$

- The extension to more than three events is similar.

Example (Cont'd)

Recall that among male high school athletes,

> 5% go on to play at the college level.
>
> 1.7% of those who play at the college level go on to play professionally.

It's also known that

> 40% of those who play in college and then play professionally have a career that lasts more than 3 years.

Let

$$\begin{aligned} A &= \text{The athlete has a career that lasts more than 3 years} \\ B &= \text{The athlete plays professionally} \\ C &= \text{The athlete plays in college} \end{aligned}$$

Then

$$P(C) = 0.05, \quad P(B|C) = 0.017, \quad \text{and} \quad P(A|B \cap C) = 0.40$$

so the probability that an athlete will play in college **and** turn pro **and** last more than 3 years is

$$\begin{aligned} P(A \cap B \cap C) &= P(C)P(B|C)P(A|B \cap C) \\ &= (0.05)(0.017)(0.40) = 0.00034. \end{aligned}$$

Independence (2.5)

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- It can be shown that the definition above implies

$$P(B|A) = P(B)$$

  too.

- The definition of independence is equivalent to the following rule.

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- In fact, the above rule is sometimes used as the *definition* of independence.
- In practice, we usually know (or assume) events are independent, then use the rule to compute $P(A \cap B)$.

Example

Here are some examples:

- Two coin tosses.

$$\begin{aligned} P(\text{Two heads}) &= P(\text{1st is head}) \times P(\text{2nd is head}) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

- Two rolls of a die.

$$\begin{aligned} P(\text{Two one's}) &= P(\text{1st is one}) \times P(\text{2nd is one}) \\ &= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}. \end{aligned}$$

- We can extend the definition of *independence* to more than two events.

  Events $A_1, A_2, \ldots, A_n$ are said to be ***mutually independent*** if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots i_k$,

  $$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \;=\; P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

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- Thus for three events A, B, and C, the definition requires that *all four* of the following be met:

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B)P(C) \\ P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

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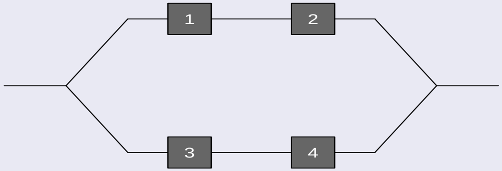
- In practice, we usually know (or assume) events are mutually independent, then use the rule to compute

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \;=\; P(A_1)P(A_2)\cdots P(A_n)$$

Example

Consider the system of electrical components connected as in the picture below.



Components 1 and 2 are connected in series, so that subsystem works only if both 1 and 2 work. Similarly for components 3 and 4 and that subsystem. The entire system works only if at least one of the subsystems works.

Let

$$A_i = \text{The } i\text{th component works.}$$

Assume the A_i's are **mutually independent** and suppose $\boldsymbol{P(A_i) = 0.9}$ for every i. Then

$$\begin{aligned}
P(\text{System works}) &= P((A_1 \cap A_2) \cup (A_3 \cap A_4)) \\
&= P(A_1 \cap A_2) + P(A_3 \cap A_4) \\
&\quad - P((A_1 \cap A_2) \cap (A_3 \cap A_4)) \\
&= (0.9)(0.9) + (0.9)(0.9) \\
&\quad -(0.9)(0.9)(0.9)(0.9) \\
&= 0.964.
\end{aligned}$$