Probability and Statistics

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Topics

Objectives

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- Recognize problems for which the Law of Total Probability is applicable
- Use the Law of Total Probability to solve probability problems
- Recognize problems for which Bayes Rule is applicable
- Use Bayes Rule to solve probability problems

Law of Total Probability (2.4)

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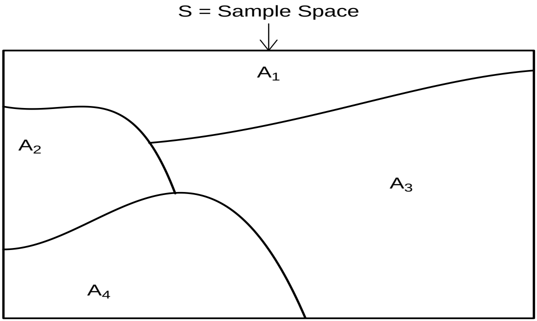
- If $A_1, A_2, \ldots, A_n$ are a partition of $\mathcal{S}$, then the outcome of the random experiment will satisfy *one and only one* of the A_i's.



$A_1 \cup A_2 \cup A_3 \cup A_4 = S$

- The ***Law of Total Probability*** says:

Law of Total Probability: If $A_1, A_2, \ldots, A_n$ are a partition of $\mathcal{S}$, and B is *any* event, then

$$P(B) \;=\; \sum_{i=1}^{n} P(B \cap A_i)$$

which, upon replacing each $P(B \cap A_i)$ by $P(A_i)P(B|A_i)$, can also be written as

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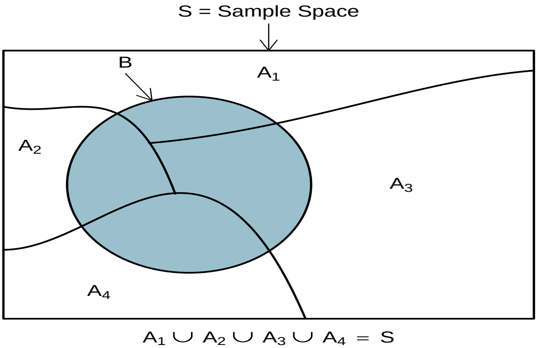
S = Sample Space
B
A_1
A_2
A_3
A_4
$A_1 \cup A_2 \cup A_3 \cup A_4 = S$

- The **Law of Total Probability** is used to find the **probability** of an event B when:

  1. The probabilities of B occurring under each of several conditions $A_1, A_2, \ldots, A_n$ are known.
  2. The probabilities of each of those conditions being met are known.

Example

A diagnostic test for a certain disease is **positive 98%** of the time when the disease is **present**, and **(falsely) positive 5%** of the time when the disease is **absent.**

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If the disease affects **1 out of every 1,000** people, what's the **probability** that a randomly selected person would test **positive** for the disease?

To answer this question, we'll apply the **Law of Total Probability**.

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We know the **probability** of a **positive** test result (B) under each of two conditions: disease **present** (A_1) and disease **absent** (A_2).

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We know the **probability** of a **positive** test result (B) under each of two conditions: disease **present** ($\boldsymbol{A_1}$) and disease **absent** ($\boldsymbol{A_2}$).

$$\begin{aligned} P(B \mid A_1) &= \mathbf{0.98} \\ P(B \mid A_2) &= \mathbf{0.05} \end{aligned}$$

We also know the **probability** of each of those two conditions (disease **present**, A_1, and **absent**, A_2).

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P(A_1) &= \frac{\mathbf{1}}{\mathbf{1,000}} \\
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Bayes Rule (2.4)

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$$\text{from} \quad P(B \mid A) \quad \text{to} \quad P(A \mid B).$$

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Examples:

- **Imperfect medical test for a disease** – If a person has a certain disease, there’s a 98% chance they’ll test positive. If a person tests positive, what’s the chance they have the disease?

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Examples:

- **Imperfect medical test for a disease** – If a person has a certain disease, there's a 98% chance they'll test positive. If a person tests positive, what's the chance they have the disease?
- **The "Prosecutor's Fallacy"** – A suspect matching the description of the perpetrator is detained. The chance that an innocent person would match the description is one in a million. What's the chance that a person matching the description is innocent?

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which is ***Bayes Rule***:

Bayes' Rule: If $A_1, A_2, \ldots, A_n$ are a partition of $\mathcal{S}$, and B is *any* event, then for each j,

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}.$$

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Suppose again that a disease affects **1 out of every 1,000** people, and a test is **positive 98%** of the time when the disease is **present**, and **(falsely) positive 5%** of the time when the disease is **absent.**

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Recall that the **probabilities** of a **positive** test result (B) under each of the conditions disease **present** (A_1) and disease **absent** (A_2) are

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i.e. only about a 2% chance!

- In **Bayes Rule** applications, $P(A_j)$ is called the ***prior probability*** of the event A_j and $P(A_j|B)$ is called the ***posterior probability***.

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  The idea is that we use the information that B has occurred to "update" the probability of the event A_j occurring.