Probability and Statistics

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Notes

Topics

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Objectives

Objectives:

- Recognize Poisson random variables.
- Compute probabilities involving a Poisson random variable.
- Compute and interpret the expected value, variance, and standard deviation of a Poisson random variable.
- Recognize a Poisson Process.
- Compute probabilities involving a Poisson Process.

Notes

Poisson Random Variables (3.6)

- ***Poisson random variables*** are **counts**, **over a given period of time**, of events that occur at random time points.

  Examples:
  - The number of meteors ("shooting stars") appearing in the night sky during one-hour period.
  - The number of automobiles passing though an intersection in a 24-hour period.
  - The number of customers arriving at a store's checkout counter in a one-hour period.

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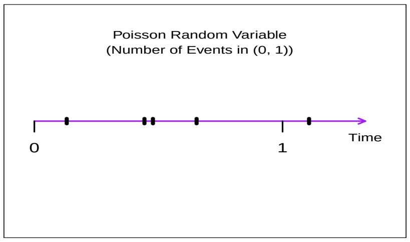

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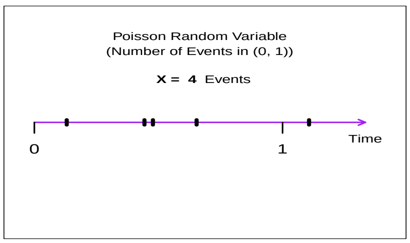

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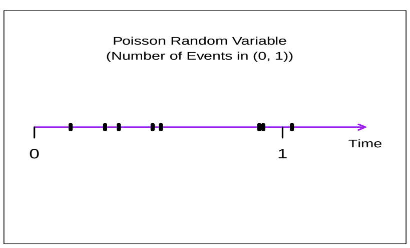

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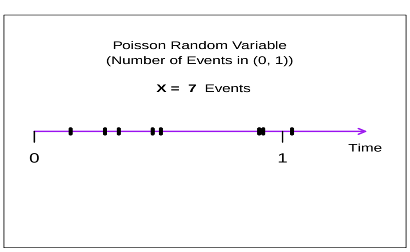

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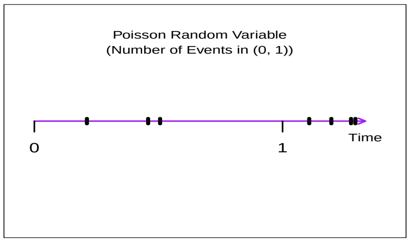

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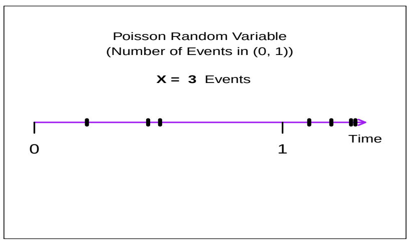

Notes

Poisson(μ) Pmf:

$$p(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad \text{for} \quad x = 0, 1, 2, \ldots$$

where $\mu > 0$.

(We'll see some intuition behind this **pmf** later.)

- The notation

  $$X \sim \text{Poisson}(\mu)$$

  means X follows a Poisson(μ) distribution.

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- Each choice of the ***parameter*** μ leads to a different Poisson distribution.

  (We'll see how to interpret μ later.)

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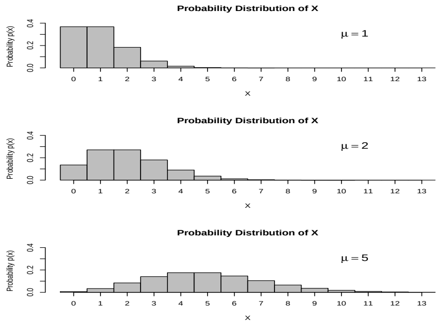

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Example

For a certain beach, let

$$X = \text{The number of visitors to the beach on a particular mid-summer day}$$

Suppose that

$$X \sim \text{Poisson}(\mathbf{10}).$$

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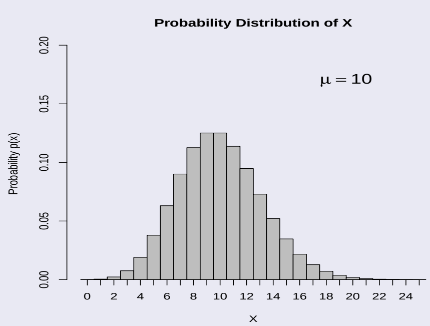

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Then the **probability** that the beach will attract **eight** visitors is

$$p(8) = \frac{10^8 e^{-10}}{8!} = \mathbf{0.1126}.$$

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The **probability** that the beach will attract anywhere **from seven to thirteen** visitors is

$$\begin{aligned} P(7 \le X \le 13) &= \sum_{x=7}^{13} p(x) \\ &= \sum_{x=7}^{13} \frac{10^x\, e^{-10}}{x!} \\ &= \frac{10^7\, e^{-10}}{7!} + \frac{10^8\, e^{-10}}{8!} + \ldots + \frac{10^{13}\, e^{-10}}{13!} \\ &= \mathbf{0.7343}. \end{aligned}$$

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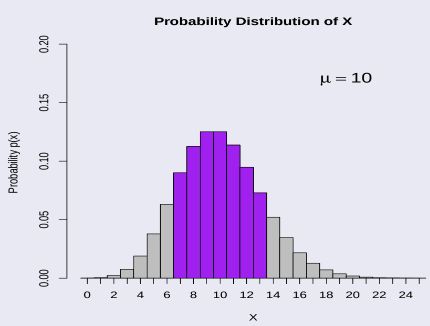

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Poisson Mean and Variance: If $X \sim$ Poisson(μ), then

$$\begin{aligned} E(X) &= \mu \\ V(X) &= \mu \end{aligned}$$

Proofs:

$$E(X) = \sum x\, p(x) = \sum x \frac{\mu^x e^{-\mu}}{x!} = \cdots = \mu.$$

$$\begin{aligned} V(X) &= \sum (x-\mu)^2\, p(x) \\ &= \sum (x-\mu)^2 \frac{\mu^x e^{-\mu}}{x!} = \cdots = \mu. \end{aligned}$$

Notes

- If X is the **number of events** occurring in a given time period, then μ is the **expected number of events**, i.e. the number that occur ***on average*** in time periods **of that length**.
- For the Poisson distribution, the **mean** and **variance** are **equal**.

  In particular, the **larger** the **mean count** is, the **more variation** there will be in the **counts**.
- **Example**: The number of visitors to a *popular* beach *varies more* from day to day than the number of visitors to an *unpopular* one.

Notes

The Poisson Distribution as a Limit (3.6)

Proposition

Suppose that in the binomial(n,p) pmf, we let $n \to \infty$ and $p \to 0$ in such a way that $np = \mu$ (for some constant μ). Then for each fixed x,

$$\binom{n}{x} p^x (1-p)^{n-x} \quad \to \quad \frac{\mu^x e^{-\mu}}{x!}.$$

- Thus in the **binomial** setting, if the number of trials $\boldsymbol{n}$ **is large** and the *success* probability $\boldsymbol{p}$ **is small**, then the number of successes X among the n trials is **approximately** a **Poisson** random variable with parameter $\boldsymbol{\mu = np}$.

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- In other words, if

$$X \sim \text{binomial}(n, p),$$

where $\boldsymbol{n}$ is **large** and $\boldsymbol{p}$ is **small**, then

$$P(X = x) \quad = \quad \binom{n}{x} p^x (1-p)^{n-x} \quad \approx \quad \frac{\mu^x\, e^{-\mu}}{x!},$$

with $\boldsymbol{\mu = np}$.

This is sometimes called the ***Poisson approximation to the binomial***.

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Example

A football stadium is filled with $\boldsymbol{n = 60,000}$ people. Suppose each person will lose his or her wallet or purse during the game with probability $\boldsymbol{p = 0.0001}$. Let

$X =$ The number of people who lose their purse or wallet

Then

$$\boldsymbol{X \sim \text{binomial}(60,000,\ \ 0.0001)}.$$

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By the **Poisson approximation to the binomial**, with $\boldsymbol{\mu = np = 6}$, the **probability** that exactly **five** people will lose their purse or wallet is

$$\begin{aligned} P(X=5) &= \binom{60,000}{5} 0.0001^5 \,(1-.0001)^{60,000-5} \\ &\approx \frac{6^5 e^{-6}}{5!} \\ &= \mathbf{0.1606231}. \end{aligned}$$

Note that the **exact binomial probability** is

$$\binom{60,000}{5} 0.0001^5 \,(1-.0001)^{60,000-5} \;=\; \mathbf{0.1606285}.$$

Notes

The Poisson Process (3.6)

- Suppose events occur at **random points in time** with a ***rate*** of α events per **one-unit of time**, on average.

  Suppose also that the value of α **doesn't change over time**.

  Let

  X = The number of events that occur in a time period that's t **units long**

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- Then under certain conditions (listed later),

$$\boldsymbol{X} \sim \textbf{Poisson}(\alpha t),$$

(Poisson with mean $\mu = \alpha t$), and so

$$P(X = x) = \frac{(\alpha t)^x e^{-\alpha t}}{x!} \qquad \text{for} \quad x = 0, 1, 2, \ldots.$$

The occurrence of events over time as just described is called a ***Poisson process*** with ***rate*** α.

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Example

According to the American Meteorological Society, meteors typically appear at a **rate** of **eight per hour** in late summer.

a) **On average**, how many meteors appear in a **half-hour** period? In a **two-hour** period?

b) Find the **probability** that exactly **five** meteors will appear **between 10:30 and 11:00** PM.

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c) Find the **probability** that **none** will appear **between 10:30 and 11:00**? Find the **probability** that **at least one** will appear.

d) Find the **probability** that exactly **15** will appear **between 10:00 and 12:00**?

Notes

- A function $f(h)$ is said to be $o(h)$ ("**little o of h**") if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0.$$

i.e. $f(h) \to 0$ even faster as $h \to 0$.

A function that's $o(h)$ can be considered to be **negligibly small** when h is **near zero**, i.e.

$$\boldsymbol{o(h) \approx 0}$$

whenever h is close to zero.

Notes

- **Poisson Process Conditions**: For the Poisson process, we require the following conditions.

  1. There exists a value $\alpha > 0$ such that for any very small time interval of length Δt, the probability that *exactly one* event occurs is $\alpha \Delta t + o(\Delta t) \approx \alpha \Delta t$.
  2. The probability of *more than one* event occurring in the time interval Δt is $o(\Delta t) \approx 0$.
  3. The number of events occurring in the time interval Δt is *independent* of the number of events that occurred prior to this time interval.

Notes

- For a **time period** $(0, t)$, let

$$X = \text{The number events in } (0, t).$$

To see **why**

$$\boldsymbol{X \sim} \textbf{ Poisson}\boldsymbol{(\alpha t)},$$

consider partitioning $(0, t)$ into very **small intervals**, each of length $\boldsymbol{\Delta t}$.

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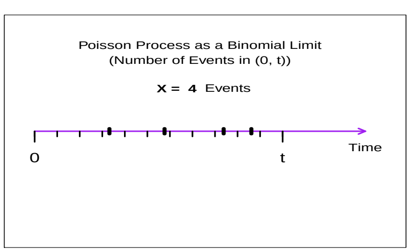

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- Then

$$X \sim \textbf{binomial}(n,\, p) \qquad \text{approximately,}$$

where

$$n = t/\Delta t \qquad \text{and} \qquad p = \alpha \Delta t.$$

Notes

- Letting $\Delta t \to 0$, we have a **binomial**(n, p) distribution with

$$n \to \infty \qquad \text{and} \qquad p \to 0$$

and

$$np = \alpha t \qquad \text{(a constant),}$$

so in the limit,

$$P(X = x) = \frac{(\alpha t)^x \, e^{-\alpha t}}{x!}.$$

i.e. in the limit,

$$X \sim \textbf{Poisson}(\alpha t).$$

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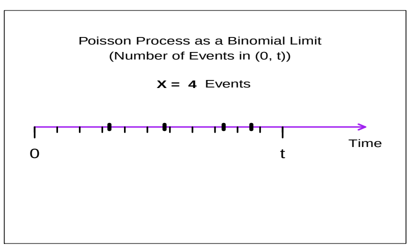

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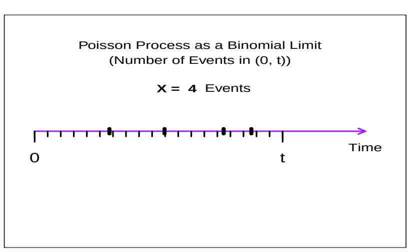

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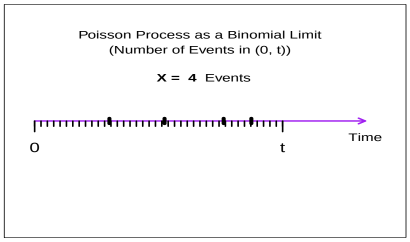

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