Probability and Statistics

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Topics

Objectives

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- Recognize Poisson random variables.
- Compute probabilities involving a Poisson random variable.
- Compute and interpret the expected value, variance, and standard deviation of a Poisson random variable.
- Recognize a Poisson Process.
- Compute probabilities involving a Poisson Process.

Poisson Random Variables (3.6)

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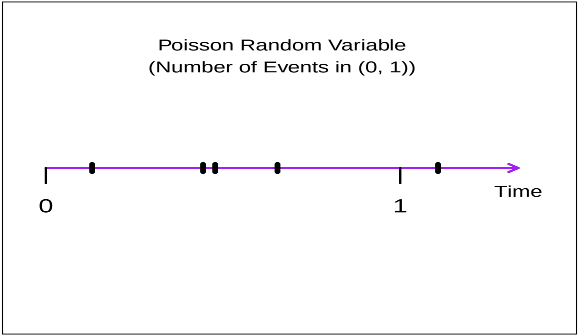
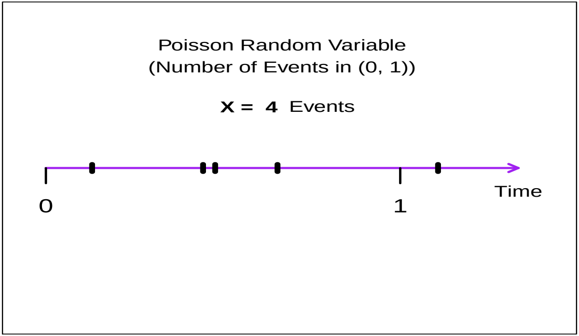
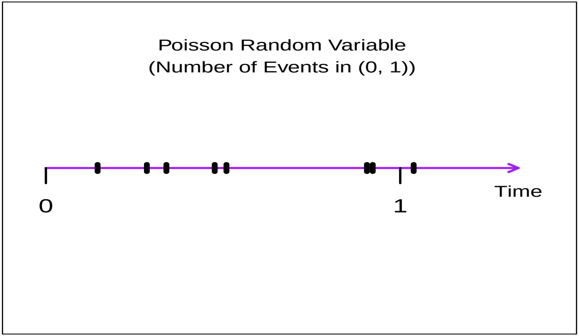
  - The number of automobiles passing though an intersection in a 24-hour period.
  - The number of customers arriving at a store’s checkout counter in a one-hour period.


Poisson Random Variable
(Number of Events in (0, 1))
0
1
Time


Poisson Random Variable
(Number of Events in (0, 1))
X = 4 Events
0
1
Time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1
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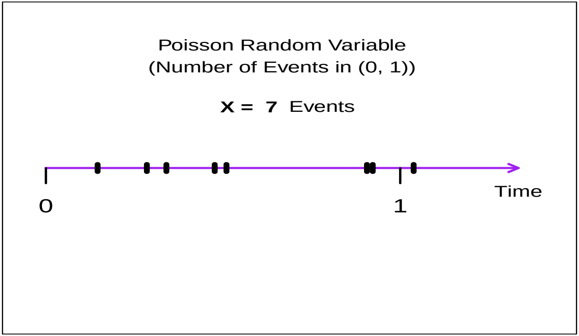
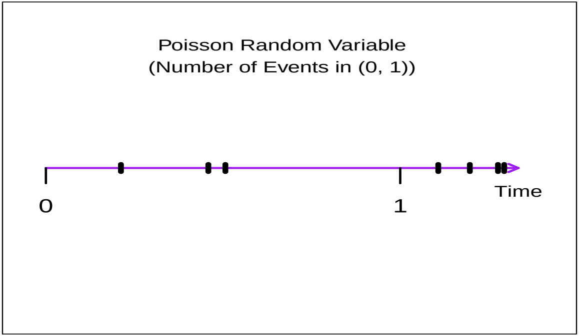
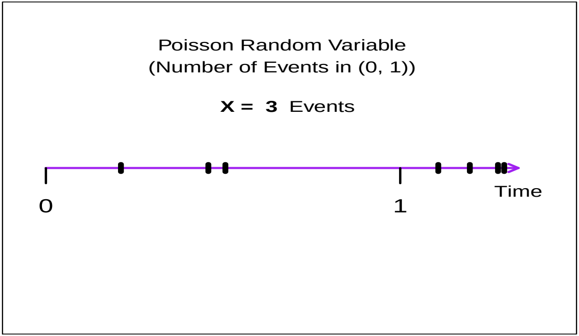

Poisson Random Variable
(Number of Events in (0, 1))
X = 7 Events
0
1
Time


Poisson Random Variable
(Number of Events in (0, 1))
0
1
Time


Poisson Random Variable
(Number of Events in (0, 1))
X = 3 Events
0
1
Time

Poisson(μ) Pmf:

$$p(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad \text{for} \quad x = 0, 1, 2, \ldots$$

where $\mu > 0$.

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- The notation

$$\boldsymbol{X} \;\sim\; \text{Poisson}(\boldsymbol{\mu})$$

  means X follows a Poisson(μ) distribution.

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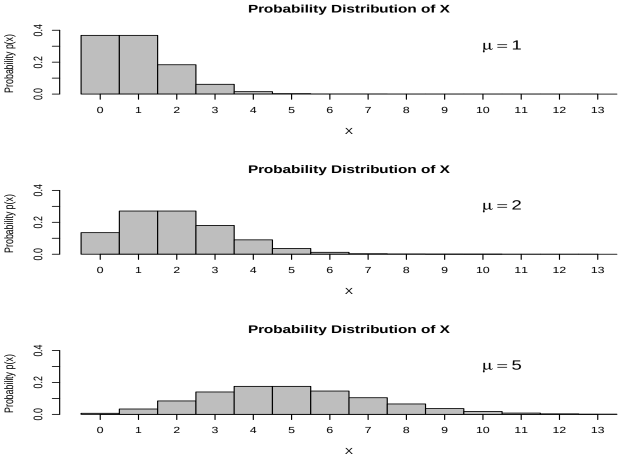
  (We'll see how to interpret μ later.)


Probability Distribution of X
μ = 1
Probability p(x)
0.0
0.2
0.4
0 1 2 3 4 5 6 7 8 9 10 11 12 13
x
Probability Distribution of X
μ = 2
Probability p(x)
0.0
0.2
0.4
0 1 2 3 4 5 6 7 8 9 10 11 12 13
x
Probability Distribution of X
μ = 5
Probability p(x)
0.0
0.2
0.4
0 1 2 3 4 5 6 7 8 9 10 11 12 13
x

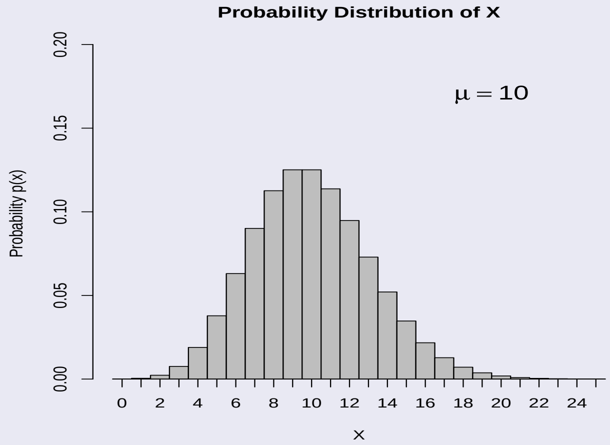
Example

For a certain beach, let

$$X = \text{The number of visitors to the beach on a particular mid-summer day}$$

Suppose that

$$X \sim \text{Poisson}(\mathbf{10}).$$


Probability Distribution of X
μ = 10
Probability p(x)
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
12
14
16
18
20
22
24
x

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$$p(8) \ = \ \frac{10^8 \, e^{-10}}{8!} \ = \ \mathbf{0.1126}.$$

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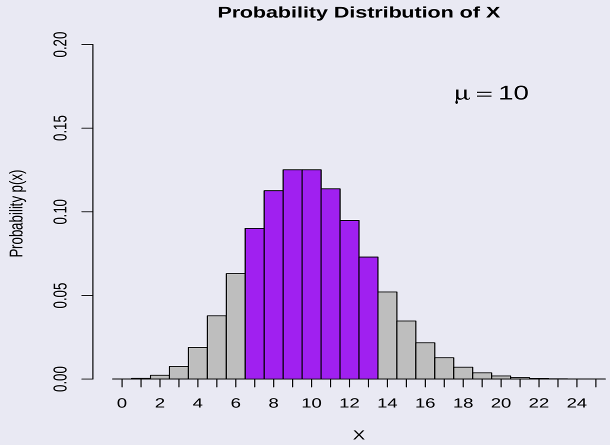
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μ = 10
Probability p(x)
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0.05
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Poisson Mean and Variance: If $X \sim$ Poisson(μ), then

$$\begin{aligned} E(X) &= \mu \\ V(X) &= \mu \end{aligned}$$

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Proofs:

$$E(X) = \sum x\, p(x) = \sum x \frac{\mu^x e^{-\mu}}{x!} = \cdots = \mu.$$

$$\begin{aligned} V(X) &= \sum (x-\mu)^2\, p(x) \\ &= \sum (x-\mu)^2 \frac{\mu^x e^{-\mu}}{x!} = \cdots = \mu. \end{aligned}$$

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  In particular, the **larger** the **mean count** is, the **more variation** there will be in the **counts**.
- **Example**: The number of visitors to a *popular* beach *varies more* from day to day than the number of visitors to an *unpopular* one.

The Poisson Distribution as a Limit (3.6)

Proposition

Suppose that in the binomial(n, p) pmf, we let $n \to \infty$ and $p \to 0$ in such a way that $np = \mu$ (for some constant μ). Then for each fixed x,

$$\binom{n}{x} p^x (1-p)^{n-x} \quad \to \quad \frac{\mu^x e^{-\mu}}{x!}.$$

- Thus in the **binomial** setting, if the number of trials $\boldsymbol{n}$ **is large** and the *success* probability $\boldsymbol{p}$ **is small**, then the number of successes X among the n trials is **approximately** a **Poisson** random variable with parameter $\boldsymbol{\mu = np}$.

- In other words, if

$$X \sim \text{binomial}(n, p),$$

where n is **large** and p is **small**, then

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \approx \frac{\mu^x \, e^{-\mu}}{x!},$$

with $\mu = np$.

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This is sometimes called the ***Poisson approximation to the binomial***.

Example

A football stadium is filled with $n = 60,000$ people. Suppose each person will lose his or her wallet or purse during the game with probability $p = 0.0001$. Let

$$X = \text{The number of people who lose their purse or wallet}$$

Then

$$\boldsymbol{X} \sim \text{binomial}(60,000, \;\; 0.0001).$$

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$$\begin{aligned} P(X=5) &= \binom{60,000}{5} 0.0001^5 \, (1-.0001)^{60,000-5} \\ &\approx \frac{6^5 e^{-6}}{5!} \\ &= \mathbf{0.1606231}. \end{aligned}$$

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Note that the **exact binomial probability** is

$$\binom{60,000}{5} 0.0001^5 \, (1-.0001)^{60,000-5} \; = \; \mathbf{0.1606285}.$$

The Poisson Process (3.6)

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  Suppose also that the value of α **doesn't change over time.**

  Let

  X = The number of events that occur in a time period that's t **units long**

- Then under certain conditions (listed later),

$$X \sim \textbf{Poisson}(\alpha t),$$

(Poisson with mean $\mu = \alpha t$), and so

$$P(X = x) = \frac{(\alpha t)^x e^{-\alpha t}}{x!} \qquad \text{for} \quad x = 0, 1, 2, \ldots.$$

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The occurrence of events over time as just described is called a ***Poisson process*** with ***rate*** α.

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According to the American Meteorological Society, meteors typically appear at a **rate** of **eight per hour** in late summer.

a) **On average**, how many meteors appear in a **half-hour** period? In a **two-hour** period?

b) Find the **probability** that exactly **five** meteors will appear **between 10:30 and 11:00** PM.

c) Find the **probability** that **none** will appear **between 10:30 and 11:00**? Find the **probability** that **at least one** will appear.

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d) Find the **probability** that exactly **15** will appear **between 10:00 and 12:00**?

- A function $f(h)$ is said to be $o(h)$ (”**little o of h**”) if

$$\lim_{h \to 0} \frac{f(h)}{h} \;=\; 0.$$

i.e. $f(h) \to 0$ even faster as $h \to 0$.

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A function that's $o(h)$ can be considered to be **negligibly small** when h is **near zero**, i.e.

$$\boldsymbol{o(h) \approx 0}$$

whenever h is close to zero.

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  2. The probability of *more than one* event occurring in the time interval Δt is $o(\Delta t) \approx 0$.
  3. The number of events occurring in the time interval Δt is *independent* of the number of events that occurred prior to this time interval.

- For a **time period** $(\mathbf{0}, \boldsymbol{t})$, let

$$X = \text{The number events in } (0, t).$$

To see **why**

$$\boldsymbol{X} \sim \textbf{Poisson}(\boldsymbol{\alpha t}),$$

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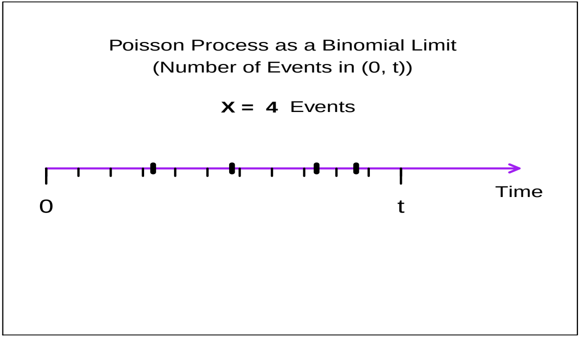
consider partitioning $(0, t)$ into very **small intervals**, each of length $\boldsymbol{\Delta t}$.


Poisson Process as a Binomial Limit
(Number of Events in (0, t))
X = 4 Events
0
t
Time

- Then

$$X \sim \textbf{binomial}(n, p) \qquad \text{approximately,}$$

where

$$n = t/\Delta t \qquad \text{and} \qquad p = \alpha\Delta t.$$

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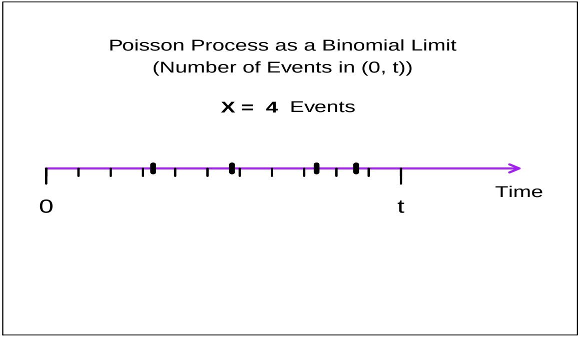
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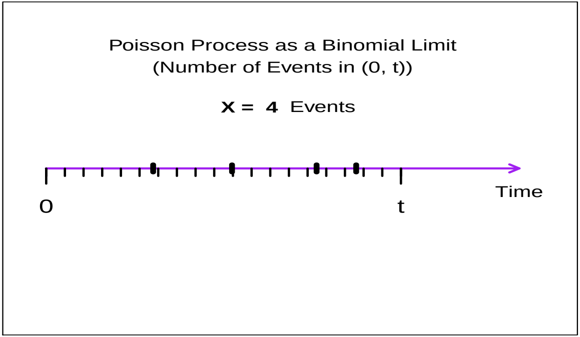
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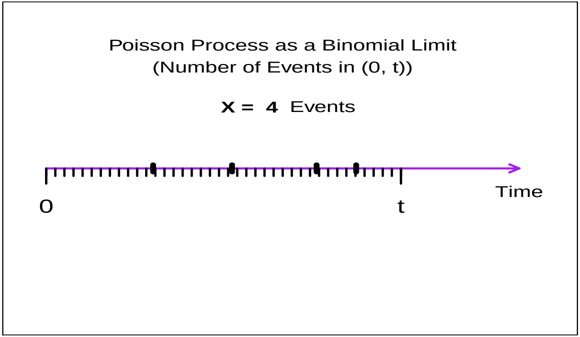
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