# Probability and Statistics

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Notes



## Topics





Notes



## Objectives

Objectives:

- Recognize exponential random variables.
- Use the exponential distribution to find probabilities.
- Find percentiles of the exponential distribution.
- State the relationship between a Poisson process and exponential random variables.
- Use the memoryless property to find exponential probabilities.



Notes



## Exponential Random Variables (4.4)

- **Exponential random variables** are used to model **waiting times** for events that occur at random time points.

  **Examples**:
  - The waiting time for a meteor ("shooting star") to appear in the night sky.
  - The waiting time for the next automobile to arrive at an intersection.
  - The waiting time for the next customer to arrive at a store's checkout counter.
- We'll see that the **memoryless property** makes exponential random variables suitable for modeling waiting times.



Notes

- The ***exponential distribution*** with **parameter $\boldsymbol{\lambda}$** has **pdf**

> **Exponential($\lambda$) Pdf**:
>
> $$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
>
> where $\lambda > 0$.

- We write

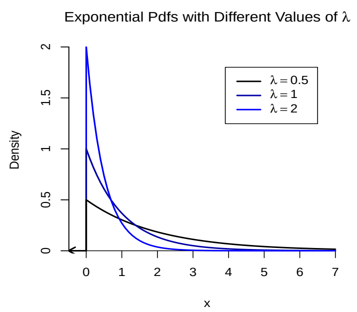
$$\boldsymbol{X \sim} \textbf{exponential}(\boldsymbol{\lambda})$$

when $X$ follows an exponential distribution.



Notes








Notes



- The mean and variance of an exponential random variable are:

> **Exponential Mean and Variance**: If $X \sim$ exponential($\lambda$) then
>
> $$E(X) = \frac{1}{\lambda}$$
> $$V(X) = \frac{1}{\lambda^2}$$



Notes



**Proofs**: To show that $E(X) = 1/\lambda$, recall that *integration by parts* says:

$$\int u\, dv\, dx = uv - \int v\, du\, dx.$$

Letting

$$u = x \quad \text{and} \quad dv = \lambda e^{-\lambda x}$$

gives

$$du = 1 \quad \text{and} \quad v = -e^{-\lambda x}$$

where $v$ was obtained from $dv$ using the *substitution rule*. So

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\ &= \int_0^{\infty} x\, \lambda e^{-\lambda x}\, dx \\ &= x\left(-e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x}\, dx \end{aligned}$$



Notes

$$= \; 0 - 0 \; - \; \left( \frac{1}{\lambda} e^{-\lambda x} \, dx \right) \Big|_0^{\infty}$$
$$= \; 0 - 0 \; - \; \left( 0 - \frac{1}{\lambda} \right)$$
$$= \; \frac{1}{\lambda}\,.$$

To show that $V(X) = 1/\lambda^2$, recall that

$$V(X) \;=\; E(X^2) \;-\; \mu^2,$$

where $\mu = E(X) = 1/\lambda$. To find $E(X^2)$, let

$$u \;=\; x^2 \qquad \text{and} \qquad dv \;=\; \lambda e^{-\lambda x}$$

so that

$$du \;=\; 2x \qquad \text{and} \qquad v \;=\; -e^{-\lambda x}\,.$$



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Now, using integration by parts,

$$E(X^2) \;=\; \int_{-\infty}^{\infty} x^2\, f(x)\, dx$$
$$=\; \int_0^{\infty} x^2\, \lambda e^{-\lambda x}\, dx$$
$$=\; x^2 \left( -e^{-\lambda x} \right) \Big|_0^{\infty} \;-\; \int_0^{\infty} -2x\, e^{-\lambda x}\, dx$$
$$=\; 0 - 0 \,+\, \int_0^{\infty} 2x\, e^{-\lambda x}\, dx\,.$$

Now use integration by parts *again* on the integral above, to get

$$E(X^2) \;=\; \frac{2}{\lambda^2}\,,$$

from which it follows that

$$V(X) \;=\; \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \;=\; \frac{1}{\lambda^2}\,.$$



Notes



- If
$$X \;\sim\; \text{exponential}(\lambda)$$
and $X$ is a random **waiting time** for an event, then
  - $E(X) = 1/\lambda$ is the mean amount of time per event.
  - $\boldsymbol{\lambda} = 1/E(X)$ is the **rate** (number of events per unit of time) .



Notes



- The **cdf** of an exponential random variable is:

**Exponential($\lambda$) Cdf**:

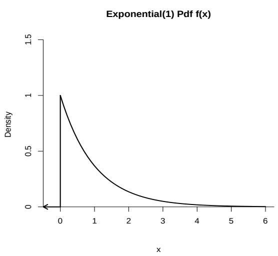
$$F(x) \;=\; \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

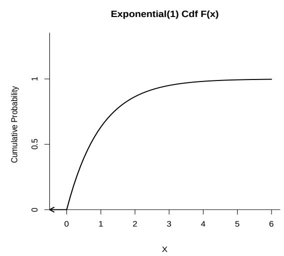
**Proof**: $F(x) = 0$ for $x < 0$. For $x \geq 0$,

$$F(x) \;=\; \int_{-\infty}^{x} f(y)\, dy$$
$$=\; \int_0^{x} \lambda e^{-\lambda y}\, dy$$
$$\vdots$$
$$=\; 1 - e^{-\lambda x}.$$



Notes



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**Example**

Suppose you're on a street corner trying to hail a taxi cab. Let

X $=$ The amount of time (in minutes) that you have to wait.

Suppose

$$\boldsymbol{X \sim} \textbf{exponential}(\mathbf{0.1})$$

Thus $\boldsymbol{\lambda = 0.1}$ (meaning the **rate** of cab arrivals is **0.1 per minute**), and so

$$E(X) \;=\; \frac{1}{0.1} \;=\; \mathbf{10} \quad \text{minutes}$$

and

$$SD(X) \;=\; \sqrt{V(X)} \;=\; \sqrt{\frac{1}{0.1^2}} \;=\; \mathbf{10} \quad \text{minutes}$$



Notes



To find the **probability** that you'll have to wait **longer than ten minutes**, either integrate the **pdf**:

$$P(X > 10) \;=\; \int_{10}^{\infty} \lambda e^{-\lambda x}\, dx$$

or just use the **cdf**:

$$\begin{aligned} P(X > 10) &= 1 - F(10) \\ &= 1 - (1 - e^{-0.1(10)}) \\ &= e^{-1} \\ &= \mathbf{0.3679}. \end{aligned}$$



Notes



To find the **probability** that you'll have to wait **between five and seven minutes**, either integrate the **pdf**:

$$P(5 < X \leq 7) \;=\; \int_{5}^{7} \lambda e^{-\lambda x}\, dx$$

or just use the **cdf**:

$$\begin{aligned} P(5 < X \leq 7) &= F(7) - F(5) \\ &= (1 - e^{-0.1(7)}) - (1 - e^{-0.1(5)}) \\ &= e^{-0.5} - e^{-0.7} \\ &= \mathbf{0.1099}. \end{aligned}$$



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To find the $50^{th}$ ***percentile*** of the distribution of $X$ (i.e. the ***median*** wait time), solve

$$F(\eta) \;=\; 0.5$$

i.e.

$$1 - e^{-0.1\eta} \;=\; 0.5$$

for $\eta$. This gives

$$\eta \;=\; -\frac{\log(0.5)}{0.1} \;=\; \mathbf{6.93} \quad \text{minutes.}$$



Notes



## Relationship to the Poisson Process (4.4)

- **Exponential** random variables are related to the **Poisson process**.

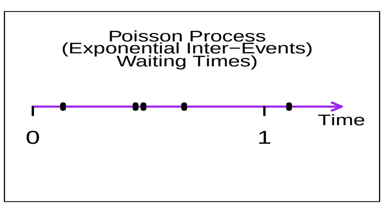
Suppose the number of events occurring in any time interval of length $t$ is a **Poisson** random variable with mean $\mu = \alpha t$ (where $\alpha$, the **rate**, is the expected number of events in one unit of time), and that the numbers of events in non-overlapping time intervals are independent of each other.

Then the elapsed time between any two successive events is an **exponential**($\lambda$) random variable with $\lambda = \alpha$.



Notes








Notes



## Memoryless Property (4.4)

- A (nonnegative) random variable $X$ is said to have the ***memoryless property*** if, for any $s > 0$ and $t > 0$,

$$P(X > s + t \,|\, X > t) \;=\; P(X > s).$$

If $X$ is a **waiting time** in minutes, say, this says that the probability that you'll need to wait **an additional** $s$ **minutes**, given that you've **already waited** $t$ **minutes**, doesn't depend on how long you've already waited ($t$).



Notes

- **Only two** kinds of random variables have the **memoryless property**.

**Memoryless Property**:

1. If $X \sim$ **geometric**$(p)$, then $X$ has the memoryless property.
2. If $X \sim$ **exponential**$(\lambda)$, then $X$ has thememoryless property.

**Proof** (for the exponential case): If

$$X \sim \text{exponential}(\lambda),$$

then (for $x \geq 0$)

$$F(x) = 1 - e^{-\lambda x}.$$



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Thus

$$\begin{aligned} P(X > s+t \,|\, X > t) &= \frac{P(\{X > s+t\} \cap \{X > t\})}{P(X > t)} \\ &= \frac{P(X > s+t)}{P(X > t)} \\ &= \frac{1 - F(s+t)}{1 - F(t)} \\ &= \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda t})} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} \\ &= 1 - F(s) \\ &= P(X > s). \end{aligned}$$



Notes



The **memoryless property** explains why **exponential random variables** they are used to model **waiting times**.

**Example**

Suppose again that you're trying to hail a taxi cab, and

X $=$ The amount of time (in minutes) that you have to wait.

Suppose again that

$$X \sim \textbf{exponential}(0.1)$$

Then the (conditional) **probability** that you'll have to wait **an additional ten minutes**, given that you've **already waited fifteen minutes**, is



Notes



$$\begin{aligned} P(X > 10 + 15 \,|\, X > 15) &= P(X > 10) \\ &= \mathbf{0.3679} \end{aligned}$$

(from a previous example).



Notes