Probability and Statistics

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Topics

Objectives

Objectives:

- Calculate and interpret:
  - CI for a normal mean μ when σ is known.
  - CI for a general population mean μ when σ is unknown and n is large.
  - CI for a normal mean μ when σ is unknown.
- Determine the sample size n for attaining a CI width under each of these three scenarios.

Introduction to Confidence Intervals (7.1)

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  In particular, the sampling error of the sample mean is

$$\textbf{Sampling Error} \;=\; \bar{X} \;-\; \mu\,.$$

- A point estimate, by itself, doesn't indicate how big the sampling error might be.

- It's preferable, therefore, to instead use an **interval estimate**, or ***confidence interval*** (or ***CI***), consisting of a range of estimates for the (unknown) population parameter.

- It's preferable, therefore, to instead use an **interval estimate**, or ***confidence interval*** (or ***CI***), consisting of a range of estimates for the (unknown) population parameter.
- The first step in computing a CI is to choose a ***level of confidence***, which measures degree of reliability of the interval.

Example

In a study of sleep deprivation among students at a university, a sample of $n = 22$ were students asked students how many hours they sleep per night.

The mean was $\bar{x} = 5.77$ hours with a standard deviation of $s = 1.57$ hours.

A **95% confidence interval** for μ, the mean number of hours slept per night in the student population, is

$$\mathbf{(5.07,\ \ 6.47)}$$

Any value of μ in this range is considered plausible, at the 95% confidence level.

CI for a Normal Mean μ When σ is Known (7.1)

- Suppose
  1. $X_1, X_2, \ldots, X_n$ are a random sample from a **normal** population.
  2. The population mean μ **is unknown** but the standard deviation σ **is known.**

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- Suppose
  1. $X_1, X_2, \ldots, X_n$ are a random sample from a **normal** population.
  2. The population mean μ **is unknown** but the standard deviation σ **is known.**
- We'll first derive a CI for μ using **level of confidence 95%.**

- We know that

$$\bar{X} \sim \mathrm{N}\left(\mu, \ \frac{\sigma}{\sqrt{n}}\right),$$

- We know that

$$\bar{X} \sim \mathsf{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right),$$

so

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathsf{N}(0, 1).$$

- Recall that $\mathbf{z_{0.025} = 1.96}$, so the area under the standard normal curve between $\mathbf{-1.96}$ and $\mathbf{1.96}$ is **0.95**.
- Therefore,

$$0.95 = P\left(-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right)$$

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&= P\left(\bar{X}+1.96\frac{\sigma}{\sqrt{n}} > \mu > \bar{X}-1.96\frac{\sigma}{\sqrt{n}}\right)
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95% Confidence Interval for a Normal Mean μ When σ is Known:

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This CI is valid when the sample is from a normal population and σ is known.

- For other levels of confidence, we replace 1.96 by the appropriate **critical value** $z_{\alpha/2}$.

100$(1-\alpha)$% Confidence Interval for a Normal Mean μ When σ is Known:

$$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

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- Commonly used levels are **90%**, **95%**, and **99%.**

Example

The National Assessment of Educational Progress Study examined quantitative skills of young adult Americans. Men aged 21 to 25 years were given a short test of their quantitative skills. Scores on the test range from 0 to 500.

In a sample of $n = 20$ young men who took the test, the sample mean score was

$$\bar{x} = 272$$

Suppose it's reasonable to assume that the distribution of scores in the population is **normal** with **known standard deviation** $\sigma = 60$, but with a mean μ whose value is **unknown**.

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Suppose it's reasonable to assume that the distribution of scores in the population is **normal** with **known standard deviation $\boldsymbol{\sigma = 60}$**, but with a mean $\boldsymbol{\mu}$ whose value is **unknown.**

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We can be **95% confident** that the true (unknown) mean $\boldsymbol{\mu}$ is in this interval somewhere.

If we use a **90% level of confidence** instead, the **critical value** is $z_{0.05} = 1.64$ and we end up with

$$(250.0, \ 294.0)$$

Note that this interval is **narrower** than the 95% interval (which was (245.7, 298.3)).

Properties of Confidence Intervals (7.1)

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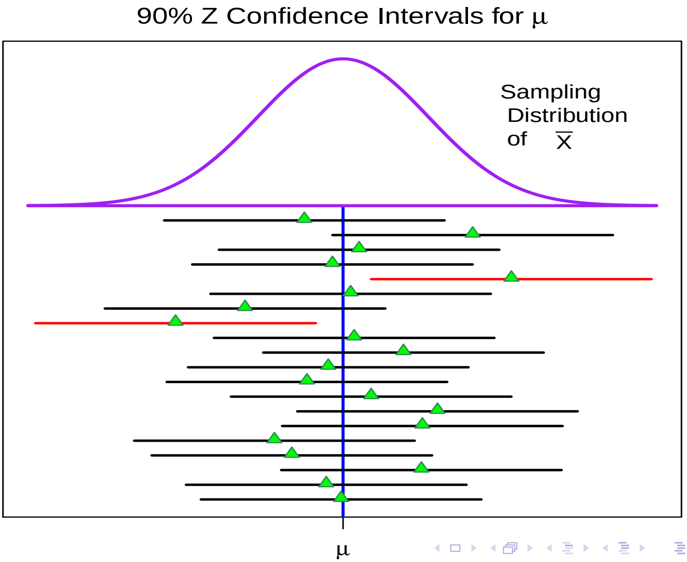
- A confidence level of 90% implies that 90% of all samples would give an interval that contains μ.


90% Z Confidence Intervals for μ
Sampling
Distribution
of X̄
μ

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- Confidence intervals have the following properties:

  1. Using a **larger sample size** n results in a **narrower** CI.
  2. A **smaller population standard deviation** σ leads to a **narrower** CI.
  3. Using a **lower level of confidence** results in a **narrower** CI.

Sample Size Determination (7.1)

- We can make the **width** of the confidence interval as **small** as we want by using a **large enough sample size** n.

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- We can make the **width** of the confidence interval as **small** as we want by using a **large enough sample size n.**
- Suppose we want the **width** to be w. Then we'd need n to be large enough that

$$2\left(z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \;=\; w.$$

Solving for n gives the required sample size.

Sample Size for a Desired CI Width: The width of the $100(1-\alpha)\%$ confidence interval for μ will be w when

$$n = \left(2\, z_{\alpha/2} \cdot \frac{\sigma}{w}\right)^2 .$$

This calculation is valid when the sample is to be from a normal population and σ is known.

Example (Cont'd)

Recall that scores on the National Assessment of Educational Progress Study quantitative skills test are a **normal** population with **known standard deviation** $\sigma = 60$, but with a mean μ whose value is **unknown.**

Suppose we want to carry out a study to estimate the population mean μ, and we want the **width** of a **95% confidence interval** for μ to be 10 units.

How big would n **need to be**?

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which we **round up** to $n = \mathbf{554}$.

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2. The **sample standard deviation S will remain fairly constant** from one sample to the next, and **approximately equal to σ.**

- As a consequence, **when n is large**,

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim \text{N}(0,\,1) \qquad \text{(approximately)}$$

even if the sample is from **non-normal** population.

(Note that σ was replaced by S above.)

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  even if the sample is from **non-normal** population.

  (Note that $\boldsymbol{\sigma}$ was replaced by S above.)

- Thus we can use the previously derived CI with $\boldsymbol{\sigma}$ **replaced by S**.

Large-Sample 100$(1-\alpha)$% Confidence Interval for a Population Mean μ:

$$\bar{X} \pm z_{\alpha/2} \frac{S}{\sqrt{n}}.$$

This CI is valid when n is large, regardless of the shape of the population distribution.

- In practice, n is **large enough** if $n > 40$.

Sample Size Determination (7.2)

- We can determine the (approximate) **sample size** n needed for the CI to have **width** w.

Sample Size for a Desired CI Width: The width of the $100(1-\alpha)\%$ confidence interval for μ will be approximately w when

$$n = \left(2\, z_{\alpha/2} \cdot \frac{S}{w}\right)^2.$$

In practice we plug in an **educated guess for** S.

This calculation is valid when the sample is to be from a general population and σ is unknown.

CI for a Normal Mean μ When σ is Unknown (7.3)

- Suppose
  1. $X_1, X_2, \ldots, X_n$ are a random sample from a **normal** population.
  2. The population mean and standard deviation **μ and σ are both unknown.**
  3. **n isn't necessarily large.**

CI for a Normal Mean μ When σ is Unknown (7.3)

- Suppose
  1. $X_1, X_2, \ldots, X_n$ are a random sample from a **normal** population.
  2. The population mean and standard deviation μ **and** σ **are both unknown.**
  3. n **isn't necessarily large.**
- To derive a CI for μ, we'll need a **new probability distribution.**

The t Distribution (7.3)

Proposition

Suppose $X_1, X_2, \ldots, X_n$ are a random sample from a **normal** population. Then the random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

follows a t ***distribution*** with $n - 1$ ***degrees of freedom***.

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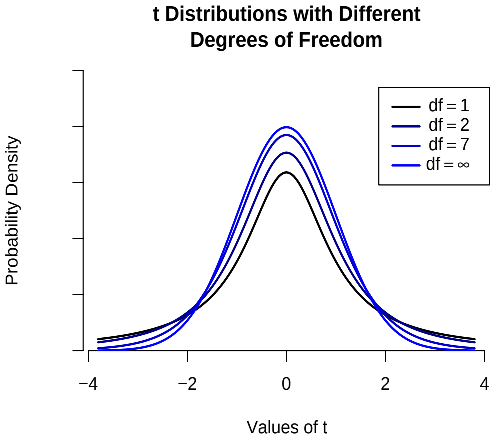
follows a t ***distribution*** with $n-1$ ***degrees of freedom***.

- We write this as

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$


t Distributions with Different
Degrees of Freedom
Probability Density
Values of t
−4
−2
0
2
4
df = 1
df = 2
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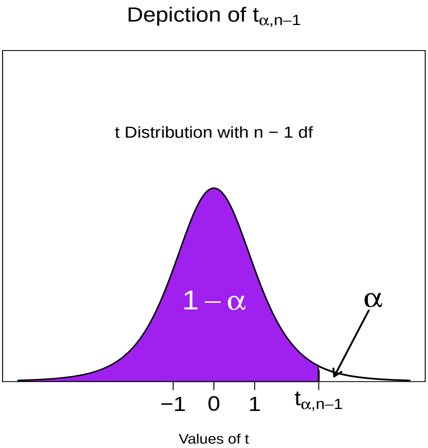
  2. It has one **parameter**, its **degrees of freedom** (or **df**).
  3. As the df increases, the t-distribution gets closer and closer to an exact N$(0,1)$ distribution.
- We use $t_{\alpha,n-1}$ to denote the t ***critical value*** that has area α to its **right** under the $t(n-1)$ curve:


Depiction of $t_{\alpha,n-1}$
t Distribution with n − 1 df
$1-\alpha$
α
−1
0
1
$t_{\alpha,n-1}$
Values of t

- Values of $t_{\alpha,n-1}$ are obtained from a t ***distribution table***.

The One-Sample t CI (7.3)

- When the sample is from a **normal** population and $\boldsymbol{\sigma}$ **is unknown**, the CI formula is derived as before, but using the $t(n-1)$ distribution instead of the **N**$(0, 1)$ distribution:

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100$(1-\alpha)$% Confidence Interval for a Normal Mean μ:

$$\bar{X} \pm t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}.$$

This is called the ***one-sample t confidence interval***.

It's valid when the sample is from a normal population and σ is unknown.

Example (Cont'd)

Recall that in the sleep deprivation study, a sample of $n = 22$ were students asked students how many hours they sleep per night.

The sample mean and standard deviation were

$$\begin{aligned} \bar{x} &= 5.77 \\ s &= 1.57 \end{aligned}$$

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The sample mean and standard deviation were

$$\begin{aligned} \bar{x} &= 5.77 \\ s &= 1.57 \end{aligned}$$

We'll estimate μ, the population mean number of hours slept per night, using **95% one-sample t CI**.

The CI is

$$\bar{X} \pm t_{0.025,21} \cdot \frac{S}{\sqrt{n}} = 5.77 \pm 2.080 \cdot \frac{1.57}{\sqrt{22}}$$

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where the value $t_{\mathbf{0.025,21}} = \mathbf{2.080}$ was obtained from a t **distribution table.**

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where the value $\mathbf{t_{0.025,21} = 2.080}$ was obtained from a $\boldsymbol{t}$ **distribution table.**

Values of μ in this interval are considered **plausible**, and we can be **95% confident** that μ is in the interval somewhere.

Sample Size Determination (7.3)

- To determine an (approximate) **sample size** for the CI to have **width** w, we use

$$n = \left(2\, z_{\alpha/2} \cdot \frac{S}{w}\right)^2$$

as before, plugging in an **educated guess for S**.

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(Note that we use $z_{\alpha/2}$, not $t_{\alpha/2,n-1}$, because $t_{\alpha/2,n-1}$ depends on n.)