

Probability and Statistics

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Topics

- 1 Introduction to Hypothesis Testing
- 2 Test for a Normal Mean μ When σ is Known

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Objectives

Objectives:

- State the null and alternative hypotheses for a given problem.
- State the role of the level of significance in hypothesis testing, and describe how the choice of a significance level can affect the conclusion of a hypothesis test.
- Interpret the p-value of a hypothesis test.
- Carry out a one-sample z test for a normal mean μ when the population standard deviation σ is known.

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Introduction to Hypothesis Testing (8.1)

- A statistical **hypothesis** is a **claim** about the value of one or more population parameters.
- A **hypothesis test** is a statistical procedure for deciding between two competing hypotheses.
- The **null hypothesis** (H_0) is the hypothesis we seek to **discredit**, but to which we give the **benefit of the doubt**.
- The **alternative hypothesis** (H_a) is the hypothesis we seek to **substantiate**.

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- The conclusion of any hypothesis test will be to either **reject** or **fail to reject** H_0 .
- The decision will be based on a **test statistic**, its associated **p-value**, and a **decision rule** involving a **level of significance**.

Steps for Carrying Out a Hypothesis Test:

1. Identify the parameter of interest.
2. Determine the null value and state H_0 and H_a .
3. Write the test statistic formula and compute its value.
4. Determine the p-value.
5. Use the decision rule, level of significance, and p-value to decide whether H_0 should be rejected, and state the conclusion.

- The **p-value** is a **probability** that answers the question:
"If H_0 was true, what's the chance we'd get a test statistic value that's as contradictory to H_0 (and consistent with H_a) as the one we got?"

Smaller p-values indicate **stronger evidence against H_0** in favor of H_a .

- The **level of significance**, denoted α , is a threshold used in the **decision rule**, which states:

Decision Rule:

Reject H_0 if p-value $< \alpha$.
Fail to reject H_0 if p-value $\geq \alpha$.

- The most commonly used values for α are **0.01**, **0.05**, and **0.10**.
- Using a **smaller** value for α means we require **stronger evidence** against H_0 before we're willing to reject H_0 .

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- When we reject H_0 , we say the result is **statistically significant**, and conclude that the result is **not just due to chance**.

Test for a Normal Mean μ When σ is Known (8.2)

- Suppose
 1. X_1, X_2, \dots, X_n are a random sample from a **normal** population.
 2. The population mean μ is **unknown** but the standard deviation σ is **known**.
- We'll see how to use the sample to decide if μ is different from some **hypothesized value** μ_0 .

The appropriate hypothesis test (when σ is known) is called the **one-sample z test for μ** .

- Because we're seeking to "disprove" the claim that μ is equal to μ_0 , the **null hypothesis** is that it *is* equal to μ_0 .

Null Hypothesis:

$$H_0 : \mu = \mu_0$$

- The **alternative hypothesis** will depend on what we're trying to "prove":

Alternative Hypothesis: The alternative hypothesis will be one of

1. $H_a : \mu > \mu_0$ (one-sided, upper-tailed)
2. $H_a : \mu < \mu_0$ (one-sided, lower-tailed)
3. $H_a : \mu \neq \mu_0$ (two-sided, two-tailed)

depending on what we're trying to verify using the data.

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Example

Hemoglobin is a protein in red blood cells that carries oxygen from the lungs to body tissues. People with less than **12** grams of hemoglobin per deciliter of blood (g/dl) are said to be anemic.

A health official in Jordan suspects that the mean μ for all children in that country is **less than 12**. To test his claim, he measures the hemoglobin a random sample of $n = 50$ children.

The **null hypothesis** is

$$H_0 : \mu = 12$$

Which of the following three **alternative hypotheses** would he test?

- $H_a : \mu > 12$
- $H_a : \mu < 12$
- $H_a : \mu \neq 12$

Would he be performing a **lower-tailed, upper-tailed, or two-tailed test**?

Example

The Survey of Study Habits and Attitudes (SSHA) is a psychological test that measures students' study habits and attitudes toward school. The mean SSHA score for all college students is **115**.

A teacher suspects that the true mean μ for *older* students is **higher than 115**. To test her claim, she gives the SSHA to a random sample of $n = 25$ students who are at least 30 years old.

The **null hypothesis** is

$$H_0 : \mu = 115$$

Which of the following three **alternative hypotheses** would she test?

- $H_a : \mu > 115$
- $H_a : \mu < 115$
- $H_a : \mu \neq 115$

Would she be performing a **lower-tailed, upper-tailed, or two-tailed test**?

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Example

The diameter of a spindle in a small motor is supposed to be 5 mm. If the spindle is either too small or too large, the motor will not work properly.

The manufacturer measures the diameters in a random sample of $n = 10$ spindles to determine whether the true mean diameter μ is **any different** from the target value 5 mm.

The **null hypothesis** is

$$H_0 : \mu = 5$$

Which of the following three **alternative hypotheses** would they test?

- $H_a : \mu > 5$
- $H_a : \mu < 5$
- $H_a : \mu \neq 5$

Would they be performing a **lower-tailed**, **upper-tailed**, or **two-tailed test**?

- The **test statistic** for the **one-sample z test for μ** is

One-Sample Z Test Statistic:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

When H_0 is true, $Z \sim N(0, 1)$.

- Z measures how many standard errors \bar{X} is away from μ_0 .
- \bar{X} is an estimator of the unknown population mean μ , so ...
 1. Z will be approximately **zero** if $\mu = \mu_0$.
 2. It will be **positive** if $\mu > \mu_0$.
 3. It will be **negative** if $\mu < \mu_0$.

1. **Large positive** values of Z provide **evidence against H_0 in favor of $H_a : \mu > \mu_0$.**
2. **Large negative** values of Z provide **evidence against H_0 in favor of $H_a : \mu < \mu_0$.**
3. **Large positive and large negative** values of Z provide **evidence against H_0 in favor of $H_a : \mu \neq \mu_0$.**

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- Recall that

$$\frac{\bar{X} - \mu\mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

- It follows that if H_0 is true (so $\mu = \mu_0$),

$$\frac{\bar{X} - \mu_0\mu_0}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Sampling Distribution of the Test Statistic Under H_0 :

If Z is the one-sample Z test statistic, then when

$$H_0 : \mu = \mu_0$$

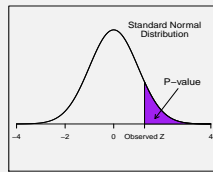
is true,

$$Z \sim N(0, 1).$$

- The **p-value** is the probability that just by chance (under H_0) we'd get a test statistic value as far from zero, in the direction predicted by H_a , as the observed value.

- P-value** = Area to the **right** of the observed z if the alternative hypothesis is $H_a : \mu > \mu_0$.

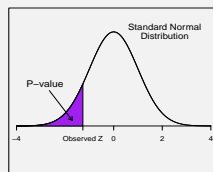
P-Value for Upper-Tailed Z Test



Values of Z

- P-value** = Area to the **left** of the observed z if the alternative hypothesis is $H_a : \mu < \mu_0$.

P-Value for Lower-Tailed Z Test



Values of Z

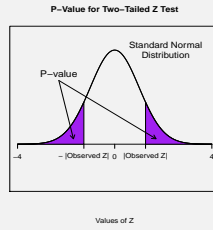
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1. **P-value** = Area to the left of $-|z|$ and right of $|z|$ if the alternative hypothesis is $H_a : \mu \neq \mu_0$.



One-Sample Z Test for μ when σ is Known

Assumptions: The data x_1, x_2, \dots, x_n are a random sample from a $N(\mu, \sigma)$ distribution where σ is known.

Null hypothesis: $H_0 : \mu = \mu_0$.

Test statistic value: $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$.

Decision rule: Reject H_0 if p-value $< \alpha$.

Alternative hypothesis

- $H_a : \mu > \mu_0$
- $H_a : \mu < \mu_0$
- $H_a : \mu \neq \mu_0$

P-value = area under $N(0, 1)$ distribution:

- to the right of z
- to the left of z
- to the left of $-|z|$ and right of $|z|$

Example

Recall that a health official in Jordan suspects that the mean hemoglobin level μ for all children in that country is **less than 12**. To test his claim, he measures the hemoglobin a random sample of $n = 50$ children.

He'll test the **hypotheses**

$$H_0 : \mu = 12$$

$$H_a : \mu < 12$$

In the **sample**, the mean hemoglobin level is

$$\bar{x} = 11.7.$$

Suppose the population standard deviation is **known** to be $\sigma = 2$.

Then the observed **test statistic** is

$$\begin{aligned} z &= \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \\ &= \frac{11.7 - 12}{2/\sqrt{50}} \\ &= -1.06. \end{aligned}$$

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Thus the **sample mean** hemoglobin level, $\bar{x} = 11.7$, is **1.06 standard errors below 12**.

The **p-value** is the **probability** that we'd get this result by chance if the **population mean μ** was **12**.

From the **left tail** of the **sampling distribution** that the test statistic would follow under H_0 (the $N(0, 1)$ distribution), the **p-value** is **0.1446**.

Thus we'd get a result like the one we got **14.46%** of the time **even if the population mean μ** was **12**.

Using a **level of significance $\alpha = 0.05$** , the **decision rule** is

Reject H_0 if p-value < 0.05 .
Fail to reject H_0 if p-value ≥ 0.05 .

Because $0.1446 \geq 0.05$, we **fail to reject H_0** .

There's **no statistically significant evidence** that the population mean hemoglobin level μ is less than 12.

The result he got (by taking a random sample) can be explained by chance variation (sampling error).

Example

A concern of employers is time spent surfing the Internet and emailing friends during work hours.

An article in the *San Luis Obispo Tribune* (Aug. 3, 2016) ran under the headline "Who Goofs Off 2 Hours a Day? Most Workers, Survey Says".

The CEO of a company wants to decide whether the average amount of wasted time per work day for her employees is **less than** the reported **120** minutes.

A random sample of $n = 10$ employees was asked about daily wasted time at work.

(They were guaranteed anonymity to obtain truthful answers!)

She'll test the **hypotheses**

$$H_0 : \mu = 120$$
$$H_a : \mu < 120$$

Here are the data:

108 112 117 122 111 131 113 113 105 128

The sample mean is

$$\bar{x} = 116.0.$$

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Suppose we know that in the company's employee population, wasted time follows a **normal** distribution with $\sigma = 9.5$.

Do the data provide **statistically significant** evidence, at the $\alpha = 0.05$ level, that the population mean wasted time μ for this company is **less than 120** minutes?

The observed **test statistic** is

$$\begin{aligned} z &= \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \\ &= \frac{116 - 120}{9.5/\sqrt{10}} \\ &= -1.33. \end{aligned}$$

Thus the **sample mean** wasted time, $\bar{x} = 116$, is **1.33 standard errors below 120**.

The **p-value** is the **probability** that we'd get this result by chance if the **population mean μ was 120**.

From the **left tail** of the **sampling distribution** that the test statistic would follow under H_0 (the $N(0, 1)$ distribution), the **p-value** is **0.0918**.

Thus we'd get a result like the one we got **9.18%** of the time **even if the population mean μ was 120**.

Using a **level of significance $\alpha = 0.05$** , the **decision rule** is

- Reject H_0 if p-value < 0.05 .
- Fail to reject H_0 if p-value ≥ 0.05 .

Because $0.0918 \geq 0.05$, we **fail to reject H_0** .

There's **no statistically significant evidence** that the population mean wasted time μ is less than 120 minutes.

The result she got (by taking a random sample) can be explained by chance variation (sampling error).

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If instead she had used a **level of significance** $\alpha = 0.10$, would the conclusion have been different?

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