# Statistical Methods

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Notes



## Topics





Notes



## Objectives

**Objectives**:

- State the equivalency between confidence intervals and hypothesis tests.
- Carry out a two-sample $t$ test for two population means.
- Compute and interpret a two-sample $t$ CI for the difference between two population means.



Notes



## Equivalency Between CIs and Hypotheses Tests

- **CIs** can be used to **test hypotheses**.

**Using CIs to Test Hypotheses**: A CI for a parameter $\theta$ with **level of confidence** $100(1-\alpha)$**%** can be used to test the hypotheses

$$\begin{aligned} H_0 : \theta &= \theta_0 \\ H_a : \theta &\neq \theta_0 \end{aligned}$$

with **significance level** $\alpha$ by invoking the following decision rule:



Notes

**Reject $H_0$** if the CI **doesn't** contain $\theta_0$
**Fail to reject $H_0$** if it **does** contain $\theta_0$

- The **CI** approach will always reach the **same conclusion** as the associated **hypothesis test**.



Notes



**Exercise**

A **95% confidence interval** for a population mean $\mu$ is

$$(21,\ 34)\,.$$

a) Could you reject the null hypothesis in a **test** of $H_0 : \mu = 23$ versus $H_a : \mu \neq 23$ at the **5% significance level**? Explain.

b) Could you reject the null hypothesis in a **test** of $H_0 : \mu = 19$ versus $H_a : \mu \neq 19$ at the **5% significance level**? Explain.



Notes



**Exercise**

For a test of

$$\begin{aligned} H_0 : \mu &= 50 \\ H_a : \mu &\neq 50 \end{aligned}$$

the **p-value** is **0.16**. Thus $H_0$ would **not be rejected** at either the **5%** or **10% significance levels**.

a) Would a **95% confidence interval** for $\mu$ contain the value **50**? Explain.

b) Would a **90% confidence interval** for $\mu$ contain the value **50**? Explain.



Notes



- To see why the **CI** approach and **hypothesis test** reach the **same conclusion**, consider a **one-sample $t$ test** of

$$\begin{aligned} H_0 : \mu &= \mu_0 \\ H_a : \mu &\neq \mu_0 \end{aligned}$$

The **rejection region approach**, with $\alpha = 0.05$, says

Reject $H_0$ if $t < -t_{0.025,\,n-1}$ or $t > t_{0.025,\,n-1}$.
Fail to reject $H_0$ otherwise.



Notes

- In other words, we **fail to reject $H_0$** if

$$t_{0.025,\,n-1} \;\le\; t \;\le\; t_{0.025,\,n-1}\,.$$

Plugging

$$t \;=\; \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

in for $t$ above, and solving for $\mu_0$, we **fail to reject $H_0$** if

$$\bar{X} \,-\, t_{0.025,\,n-1}\frac{S}{\sqrt{n}} \;\le\; \mu_0 \;\le\; \bar{X} \,+\, t_{0.025,\,n-1}\frac{S}{\sqrt{n}}\,,$$

i.e. if $\mu_0$ is in the **CI**.



Notes



## Two-Sample $t$ Test for Two Population Means $\mu_1$ and $\mu_2$ (9.1, 9.2)

- Suppose we have random samples of sizes $m$ and $n$ from **two populations**.
- We'll see how to use the samples to decide if the **population means $\mu_1$ and $\mu_2$** are different.

  The appropriate test is called the ***two-sample $t$ test for $\mu_1 - \mu_2$***.



Notes



- The **null hypothesis** is that no difference between the population means $\mu_1$ and $\mu_2$:

**Null Hypothesis**:

$$H_0 : \mu_1 - \mu_2 \;=\; 0$$



Notes



- The **alternative hypothesis** will depend on what we're trying to "prove":

**Alternative Hypothesis**: The alternative hypothesis will be one of

1. $H_a : \mu_1 - \mu_2 > 0$ (**one-sided, upper-tailed**)
2. $H_a : \mu_1 - \mu_2 < 0$ (**one-sided, lower-tailed**)
3. $H_a : \mu_1 - \mu_2 \neq 0$ (**two-sided, two-tailed**)

depending on what we're trying to verify using the data.



Notes

### The Sampling Distribution of $\bar{X} - \bar{Y}$

- Suppose $X_1, X_2, \ldots, X_m$ are a random sample from a **population** whose **mean** is $\mu_1$ and $Y_1, Y_2, \ldots, Y_n$ are random sample from a **population** whose **mean** is $\mu_2$.
- The difference $\bar{X} - \bar{Y}$ between the two **sample means** is an **estimator** of $\mu_1 - \mu_2$.



Notes



**Proposition**

If $X_1, X_2, \ldots, X_m$ are a random sample from a $N(\mu_1, \sigma_1)$ distribution and $Y_1, Y_2, \ldots, Y_n$ are random sample from a $N(\mu_2, \sigma_2)$ distribution, and the two samples are drawn *independently* of each other. Then

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right). \tag{1}$$

It follows that

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/m + \sigma_2^2/n}} \sim N(0, 1). \tag{2}$$



Notes



Furthermore, (1) and (2) still hold (at least approximately) even if the samples are from **non-normal** populations as long as the sample sizes $m$ and $n$ are both **large**.



Notes



- This follows because

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma_1}{\sqrt{m}}\right) \qquad \text{and} \qquad \bar{Y} \sim N\left(\mu_2, \frac{\sigma_2}{\sqrt{n}}\right),$$

and so $\bar{X} - \bar{Y}$ is a linear combination of two independent normal random variables.



Notes

**Proposition**

Under the assumptions stated in the last proposition, the random variable

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_1^2/m + S_2^2/n}} \sim t(\nu)$$

(at least approximately), where $S_1$ and $S_2$ are the **sample standard deviations** and the **df** $\nu$ are given by

$$\nu = \frac{(\frac{s_1^2}{m} + \frac{s_2^2}{n})^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}, \quad (3)$$

which should be rounded *down* to the nearest integer.



Notes



**Two-Sample $t$ Test Statistic for $\mu_1 - \mu_2$:**

$$T = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{S_1^2/m + S_2^2/n}}$$

- $t$ measures how many standard errors $\bar{X} - \bar{Y}$ is away from $\mathbf{0}$.
- $\bar{X} - \bar{Y}$ is an estimator of the unknown difference $\mu_1 - \mu_2$, so ...
  1. $t$ will be approximately **zero** (most likely) if $\mu_1 - \mu_2 = 0$.
  2. It will be **positive** (most likely) if $\mu_1 - \mu_2 > 0$.
  3. It will be **negative** (most likely) if $\mu_1 - \mu_2 < 0$.



Notes



1. **Large positive** values of $t$ provide **evidence against $H_0$ in favor of**
   $H_a : \mu_1 - \mu_2 > 0$.
2. **Large negative** values of $t$ provide **evidence against $H_0$ in favor of**
   $H_a : \mu_1 - \mu_2 < 0$.
3. **Large positive *and* large negative** values of $t$ provide **evidence against $H_0$ in favor of**
   $H_a : \mu_1 - \mu_2 \neq 0$.



Notes



**Sampling Distribution of the Test Statistic Under $H_0$:**
If $t$ is the two-sample $t$ test statistic, then when

$$H_0 : \mu_1 - \mu_2 = 0$$

is true,

$$t \sim t(\nu).$$



Notes

- The $t(\nu)$ curve gives us:
  - The ***rejection region*** as the **extreme 100$\alpha$% of $t$ values** (in the direction(s) specified by $H_a$).
  - The ***p-value*** as the **tail area(s) beyond the observed $t$ value** (in the direction(s) specified by $H_a$).

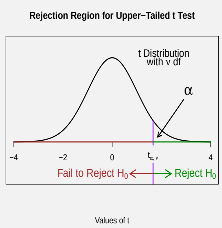
Notes





**Rejection Region**: The **rejection region** is the **set of $t$ values** in the tail of the $t(\nu)$ curve:

1. To the **right of** $t_{\alpha,\nu}$ if the alternative hypothesis is $\boldsymbol{H_a : \mu > \mu_0}$:

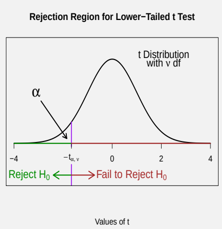



Notes





2. To the **left of** $-t_{\alpha,\nu}$ if the alternative hypothesis is $\boldsymbol{H_a : \mu < \mu_0}$:

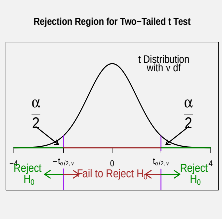



Notes





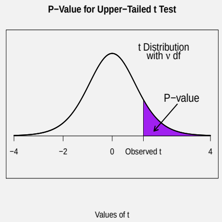
3. To the **left of $-t_{\alpha/2,\nu}$ and right of $t_{\alpha/2,\nu}$** if the alternative hypothesis is $\boldsymbol{H_a : \mu \neq \mu_0}$:




Notes

**P-Value**: The **p-value** is the **tail area** under the $t(\nu)$ curve:

1. To the **right** of the **observed** $t$ if the alternative hypothesis is $H_a : \mu > \mu_0$:

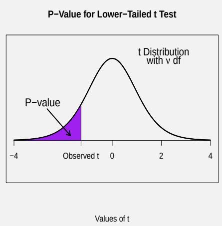





Notes



2. To the **left** of the **observed** $t$ if the alternative hypothesis is $H_a : \mu < \mu_0$:

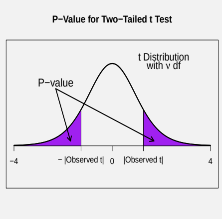





Notes



3. To the **left of** $-|\,t\,|$ **and right of** $|\,t\,|$ if the alternative hypothesis is $H_a : \mu \neq \mu_0$:






Notes



**Exercise**

Angry passengers, congested streets, time schedules, and noise and air pollution can lead to stress and premature retirement in urban bus drivers.

An intervention program designed by the Stockholm Transit District was implemented to improve the work conditions of the city's bus drivers.

Drivers were assigned to **improved** routes (**intervention**) or **normal** routes (**control**), and various physiological and psychological data were recorded for each driver.



Notes

Shown below are the data on the heart rates, in beats per minute:

| **Intervention** | | **Control** | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 68 | 66 | 74 | 52 | 67 | 63 | 77 | 57 | 80 |
| 74 | 58 | 77 | 53 | 76 | 54 | 73 | 54 | 60 |
| 69 | 63 | 77 | 63 | 60 | 68 | 64 | 66 | 71 |
| 68 | 73 | 66 | 55 | 71 | 84 | 63 | 73 | 59 |
| 64 | 76 | 68 | 64 | 82 | | | | |



Notes



Here are the summary statistics:

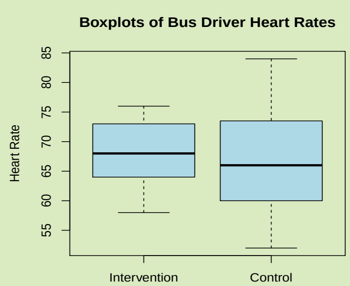
| Intervention | Control |
|---|---|
| $m = 10$ | $n = 31$ |
| $\bar{x} = 67.90$ | $\bar{y} = 66.81$ |
| $s_1 = 5.49$ | $s_2 = 9.04$ |

Side-by-side boxplots are shown on the next slide.



Notes








Notes



a) Carry out a **two-sample $t$ test** to decide if the intervention program **reduces the mean heart rate** of urban bus drivers in Stockholm. Use a **significance level** $\alpha = 0.05$.

**Hint**: The **df** are

$$\nu = \frac{\left(\frac{5.49^2}{10} + \frac{9.04^2}{31}\right)^2}{\frac{(5.49^2/10)^2}{10-1} + \frac{(9.04^2/31)^2}{31-1}} = 25.7,$$

which we round *down* to **25**, the **test statistic** ends up being $t = 0.46$ and the **p-value 0.675** (from R).

b) Can you provide an explanation for the surprising results of the study?



Notes

- **Comment**: Sometimes we want to test

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

for some (non-zero) value $\Delta_0$. In this case, $H_a$ also has $\Delta_0$ in place of 0, and the test statistic is

$$T = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{S_1^2/m + S_2^2/n}}.$$

P-values and rejection regions are exactly as described for the usual two-sample $t$ test of

$$H_0 : \mu_1 - \mu_2 = 0.$$



Notes



## Two-Sample $t$ Confidence Interval for $\mu_1 - \mu_2$

- The difference $\mu_1 - \mu_2$ is sometimes called the ***effect size***, and its estimate $\bar{X} - \bar{Y}$ the ***estimated effect size***.



Notes



**Two-Sample $t$ CI**: Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations whose means are $\mu_1$ and $\mu_2$. Then a $100(1-\alpha)\%$ ***two-sample $t$ confidence interval for $\mu_1 - \mu_2$*** is

$$\bar{X} - \bar{Y} \pm t_{\alpha/2,\,\nu} \cdot \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}, \tag{4}$$

where the **df** $\nu$ is given by (3).



Notes



- The CI is valid if either the samples are from **normal** populations or $m$ and $n$ **are large**.
- In either case, we can be $100(1-\alpha)\%$ confident that $\mu_1 - \mu_2$ will be contained in the CI.



Notes

### Exercise

Consider again the study on bus driver heart rates.

a) Give a (point) **estimate** for the **effect size** $\mu_1 - \mu_2$ of the intervention on mean heart rates.

b) Compute and interpret a **95% CI** for $\mu_1 - \mu_2$.

**Hints**: The **df** are **25** (again) and the $t$ **critical value** is $t_{0.025,\,25} = 2.060$.

c) Is **0** contained in the CI? What does that indicate about effect of the intervention on heart rates?



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