Statistical Methods

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Topics

1. Equivalency Between CIs and Hypothesis Tests
2. Two-Sample t Test for Two Population Means μ_1 and μ_2
3. Two-Sample t Confidence Interval for $\mu_1 - \mu_2$

Objectives

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- State the equivalency between confidence intervals and hypothesis tests.
- Carry out a two-sample t test for two population means.
- Compute and interpret a two-sample t CI for the difference between two population means.

Equivalency Between CIs and Hypotheses Tests

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Using CIs to Test Hypotheses: A CI for a parameter θ with **level of confidence** $100(1-\alpha)$**%** can be used to test the hypotheses

$$\begin{aligned} H_0: \theta &= \theta_0 \\ H_a: \theta &\neq \theta_0 \end{aligned}$$

with **significance level** α by invoking the following decision rule:

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Fail to reject H_0 if it **does** contain θ_0

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- The **CI** approach will always reach the **same conclusion** as the associated **hypothesis test.**

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A **95% confidence interval** for a population mean μ is

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b) Could you reject the null hypothesis in a **test** of $H_0 : \mu = 19$ versus $H_a : \mu \neq 19$ at the **5% significance level**? Explain.

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For a test of

$$\begin{aligned} H_0 : \boldsymbol{\mu} &= \mathbf{50} \\ H_a : \boldsymbol{\mu} &\neq \mathbf{50} \end{aligned}$$

the **p-value** is **0.16.** Thus H_0 would **not be rejected** at either the **5%** or **10% significance levels**.

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a) Would a **95% confidence interval** for μ contain the value **50**? Explain.

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a) Would a **95% confidence interval** for μ contain the value **50**? Explain.

b) Would a **90% confidence interval** for μ contain the value **50**? Explain.

- To see why the **CI** approach and **hypothesis test** reach the **same conclusion**, consider a **one-sample t test** of

$$\begin{aligned} H_0 : \mu &= \mu_0 \\ H_a : \mu &\neq \mu_0 \end{aligned}$$

The **rejection region approach**, with $\alpha = 0.05$, says

Reject H_0 if $t < -t_{0.025,\, n-1}$ or $t > t_{0.025,\, n-1}$.
Fail to reject H_0 otherwise.

- In other words, we **fail to reject H_0** if

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$$\bar{X} \;-\; t_{0.025,\, n-1}\frac{S}{\sqrt{n}} \;\le\; \mu_0 \;\le\; \bar{X} \;+\; t_{0.025,\, n-1}\frac{S}{\sqrt{n}}\,,$$

- In other words, we **fail to reject H_0** if

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i.e. if μ_0 is in the **CI.**

Two-Sample t Test for Two Population Means μ_1 and μ_2 (9.1, 9.2)

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Two-Sample t Test for Two Population Means μ_1 and μ_2 (9.1, 9.2)

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- We'll see how to use the samples to decide if the **population means** μ_1 and μ_2 are different.

  The appropriate test is called the ***two-sample*** t ***test for*** $\mu_1 - \mu_2$.

- The **null hypothesis** is that no difference between the population means μ_1 and μ_2:

Null Hypothesis:

$$H_0 : \mu_1 - \mu_2 \;=\; 0$$

- The **alternative hypothesis** will depend on what we're trying to "prove":

Alternative Hypothesis: The alternative hypothesis will be one of

1. $H_a : \mu_1 - \mu_2 > 0$ (**one-sided, upper-tailed**)
2. $H_a : \mu_1 - \mu_2 < 0$ (**one-sided, lower-tailed**)
3. $H_a : \mu_1 - \mu_2 \neq 0$ (**two-sided, two-tailed**)

depending on what we're trying to verify using the data.

The Sampling Distribution of $\bar{X} - \bar{Y}$

- Suppose $X_1, X_2, \ldots, X_m$ are a random sample from a **population** whose **mean** is $\boldsymbol{\mu_1}$ and $Y_1, Y_2, \ldots, Y_n$ are random sample from a **population** whose **mean** is $\boldsymbol{\mu_2}$.

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- The difference $\bar{X} - \bar{Y}$ between the two **sample means** is an **estimator** of $\mu_1 - \mu_2$.

Proposition

If $X_1, X_2, \ldots, X_m$ are a random sample from a $N(\mu_1, \sigma_1)$ distribution and $Y_1, Y_2, \ldots, Y_n$ are random sample from a $N(\mu_2, \sigma_2)$ distribution, and the two samples are drawn *independently* of each other. Then

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2,\ \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right). \qquad (1)$$

It follows that

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/m + \sigma_2^2/n}} \sim N(0, 1). \qquad (2)$$

Furthermore, (1) and (2) still hold (at least approximately) even if the samples are from **non-normal** populations as long as the sample sizes m and n are both **large**.

- This follows because

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma_1}{\sqrt{m}}\right) \qquad \text{and} \qquad \bar{Y} \sim N\left(\mu_2, \frac{\sigma_2}{\sqrt{n}}\right),$$

and so $\bar{X} - \bar{Y}$ is a linear combination of two independent normal random variables.

Proposition

Under the assumptions stated in the last proposition, the random variable

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_1^2/m + S_2^2/n}} \sim t(\nu)$$

(at least approximately), where S_1 and S_2 are the **sample standard deviations** and the **df** ν are given by

$$\nu = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}, \tag{3}$$

which should be rounded *down* to the nearest integer.

Two-Sample t Test Statistic for $\mu_1 - \mu_2$:

$$T = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{S_1^2/m + S_2^2/n}}$$

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- $\bar{X} - \bar{Y}$ is an estimator of the unknown difference $\boldsymbol{\mu_1 - \mu_2}$, so ...
    1. t will be approximately **zero** (most likely) if $\mu_1 - \mu_2 = 0$.
    2. It will be **positive** (most likely) if $\mu_1 - \mu_2 > 0$.
    3. It will be **negative** (most likely) if $\mu_1 - \mu_2 < 0$.

1. **Large positive** values of t provide **evidence against H_0 in favor of**
   $H_a : \mu_1 - \mu_2 > 0$.
2. **Large negative** values of t provide **evidence against H_0 in favor of**
   $H_a : \mu_1 - \mu_2 < 0$.
3. **Large positive *and* large negative** values of t provide **evidence against H_0 in favor of**
   $H_a : \mu_1 - \mu_2 \neq 0$.

Sampling Distribution of the Test Statistic Under H_0:
If t is the two-sample t test statistic, then when

$$H_0 : \mu_1 - \mu_2 = 0$$

is true,

$$t \sim t(\nu).$$

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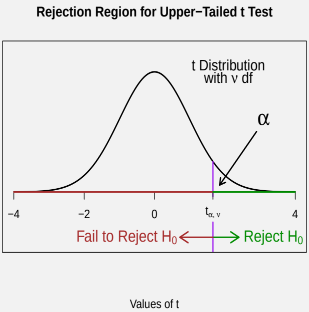
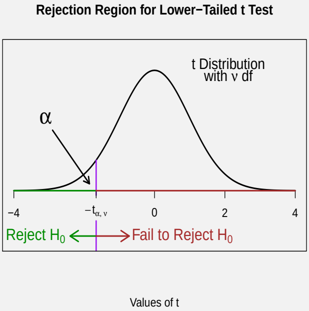
  - The ***rejection region*** as the **extreme 100α% of t values** (in the direction(s) specified by H_a).
  - The ***p-value*** as the **tail area(s) beyond the observed t value** (in the direction(s) specified by H_a).

Rejection Region: The **rejection region** is the **set of t values** in the tail of the $t(\nu)$ curve:

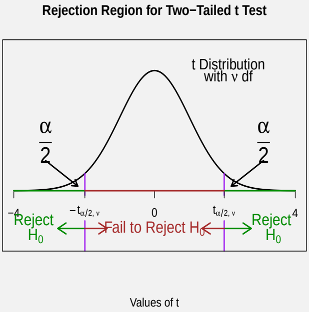
1. To the **right of** $t_{\alpha,\nu}$ if the alternative hypothesis is $H_a : \mu > \mu_0$:



2. To the **left of** $-t_{\alpha,\nu}$ if the alternative hypothesis is $H_a : \mu < \mu_0$:

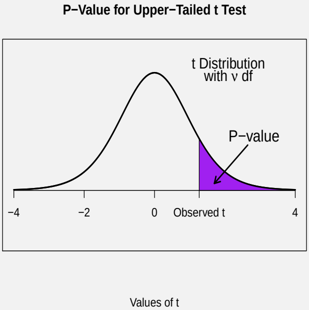


3. To the **left of** $-t_{\alpha/2,\nu}$ **and right of** $t_{\alpha/2,\nu}$ if the alternative hypothesis is $H_a : \mu \neq \mu_0$:

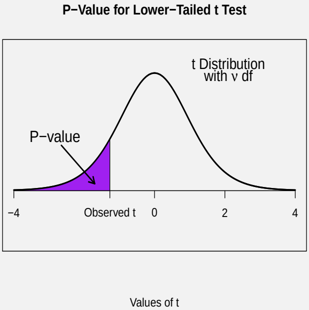


P-Value: The **p-value** is the **tail area** under the $t(\nu)$ curve:

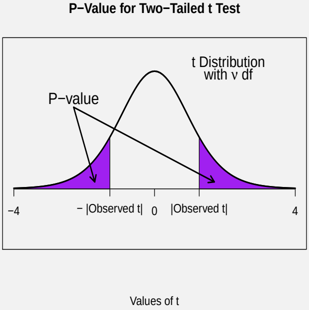
1. To the **right** of the **observed** t if the alternative hypothesis is $H_a : \mu > \mu_0$:



2. To the **left** of the **observed** t if the alternative hypothesis is $H_a : \mu < \mu_0$:



3. To the **left of** $-|t|$ **and right of** $|t|$ if the alternative hypothesis is $H_a: \mu \neq \mu_0$:



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An intervention program designed by the Stockholm Transit District was implemented to improve the work conditions of the city's bus drivers.

Drivers were assigned to **improved** routes (**intervention**) or **normal** routes (**control**), and various physiological and psychological data were recorded for each driver.

Shown below are the data on the heart rates, in beats per minute:

Intervention		**Control**						
68	66	74	52	67	63	77	57	80
74	58	77	53	76	54	73	54	60
69	63	77	63	60	68	64	66	71
68	73	66	55	71	84	63	73	59
64	76	68	64	82				

Here are the summary statistics:

Intervention			Control		
m	$=$	10	n	$=$	31
$\bar{x}$	$=$	67.90	$\bar{y}$	$=$	66.81
s_1	$=$	5.49	s_2	$=$	9.04

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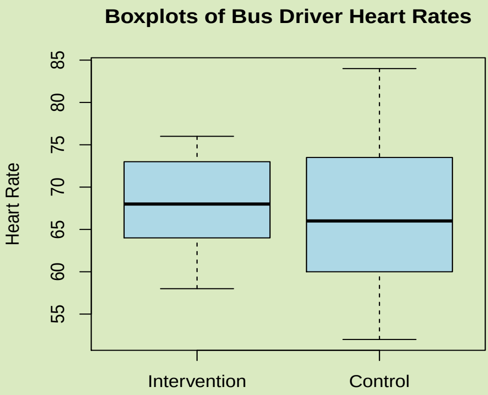
Intervention	Control
$m = 10$	$n = 31$
$\bar{x} = 67.90$	$\bar{y} = 66.81$
$s_1 = 5.49$	$s_2 = 9.04$

Side-by-side boxplots are shown on the next slide.


Boxplots of Bus Driver Heart Rates
Heart Rate
55
60
65
70
75
80
85
Intervention
Control

a) Carry out a **two-sample t test** to decide if the intervention program **reduces the mean heart rate** of urban bus drivers in Stockholm. Use a **significance level** $\alpha = 0.05$.

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Hint: The **df** are

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which we round *down* to **25**, the **test statistic** ends up being $t = 0.46$ and the **p-value 0.675** (from R).

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b) Can you provide an explanation for the surprising results of the study?

- **Comment**: Sometimes we want to test

$$H_0 : \mu_1 - \mu_2 = \Delta_0$$

for some (non-zero) value Δ_0. In this case, H_a also has Δ_0 in place of 0, and the test statistic is

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P-values and rejection regions are exactly as described for the usual two-sample t test of

$$H_0 : \mu_1 - \mu_2 = 0.$$

Two-Sample t Confidence Interval for $\mu_1 - \mu_2$

- The difference $\mu_1 - \mu_2$ is sometimes called the ***effect size***, and its estimate $\bar{X} - \bar{Y}$ the ***estimated effect size***.

Two-Sample t CI: Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations whose means are μ_1 and μ_2. Then a $100(1-\alpha)\%$ ***two-sample t confidence interval for $\mu_1 - \mu_2$*** is

$$\bar{X} - \bar{Y} \pm t_{\alpha/2,\,\nu} \cdot \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}, \tag{4}$$

where the **df** ν is given by (3).

- The CI is valid if either the samples are from **normal** populations or m and n **are large.**

- The CI is valid if either the samples are from **normal** populations or m and n **are large**.
- In either case, we can be $100(1-\alpha)\%$ confident that $\mu_1 - \mu_2$ will be contained in the CI.

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Hints: The **df** are **25** (again) and the t **critical value** is $t_{0.025,\,25} = 2.060$.

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Hints: The **df** are **25** (again) and the t **critical value** is $t_{0.025,\,25} = 2.060$.

c) Is **0** contained in the CI? What does that indicate about effect of the intervention on heart rates?