Statistical Methods

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Topics

Objectives

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- Use normal probability plots to assess whether a sample is from a normal population.
- Interpret sums of squares, degrees of freedom, and mean squares in a one-factor ANOVA context.
- State the ANOVA partition of the total variation in a data set.
- Carry out a one-factor ANOVA F test for population means $\mu_1, \mu_2, \ldots, \mu_I$.

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  Thus $X_{(1)}$ is the smallest value in the data set, $X_{(2)}$ is the second smallest, etc.

- The proposition ahead gives the **expected values** of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ **when** the sample is from a **normal** population.

Proposition

If $X_1, X_2, \ldots, X_n$ are a random sample from a N(μ, σ) distribution, then

$$\begin{aligned} E(X_{(i)}) &\approx 100p_i\text{th percentile of the } N(\mu,\sigma) \text{ distribution} \\ &= \mu + z_i\sigma \,, \end{aligned} \tag{1}$$

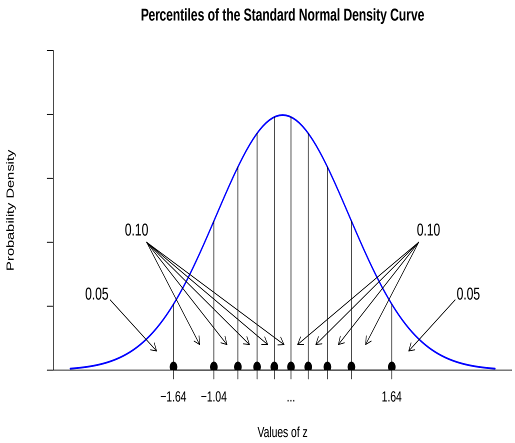
where

$$p_i = \frac{i - 0.5}{n}$$

and

$$z_i = \text{The } 100p_i\text{th percentile of the } N(0,1) \text{ distribution.}$$

- For example, in a sample of size $n = 10$ from a N$(0,\ 1)$ distribution, the **expected** sample values are the points shown on the next slide.



These points are the 5th, 15th, ..., 95th percentiles of the $\mathrm{N}(0,\ 1)$ distribution:

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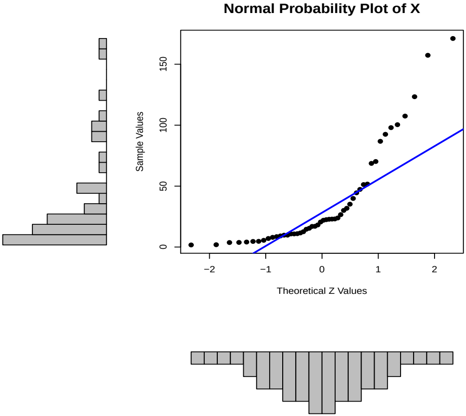
- **Curved** patterns indicate **non-normality**.


Normal Probability Plot of X
Sample Values
0
50
100
150
−2
−1
0
1
2
Theoretical Z Values

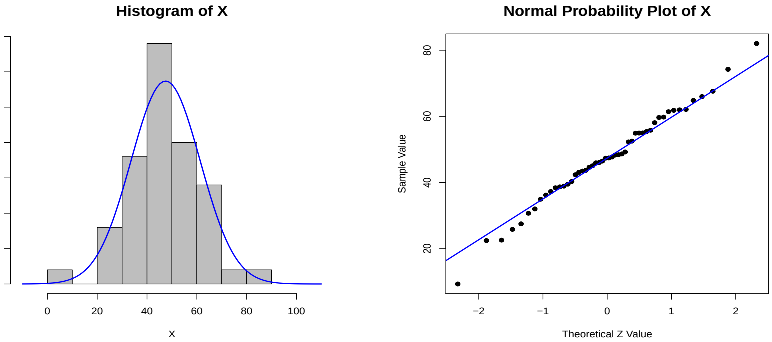


Figure: Histogram of symmetric, approximately normal data (left). Normal probability plot of the same data (right).

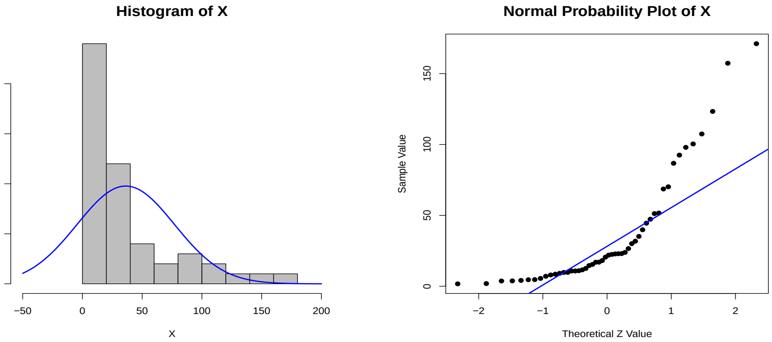


Figure: Histogram of non-normal, right skewed data (left). Normal probability plot of the same data (right).

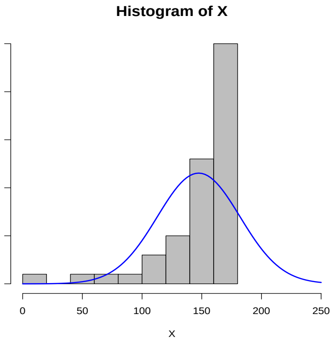


Figure: Histogram of non-normal, left skewed data (left). Normal probability plot of the same data (right).

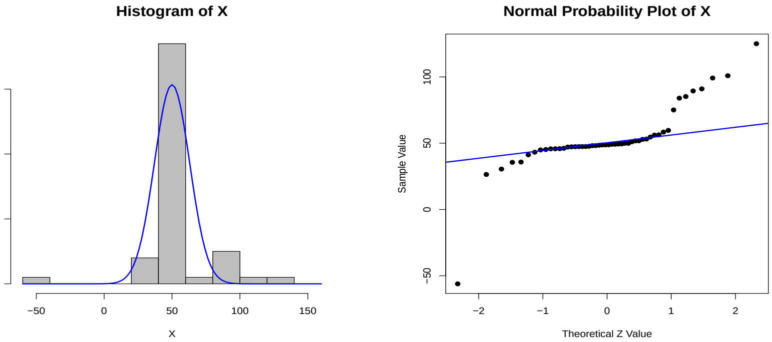


Figure: Histogram of non-normal, "heavy tailed" data (left). Normal probability plot of the same data (right).

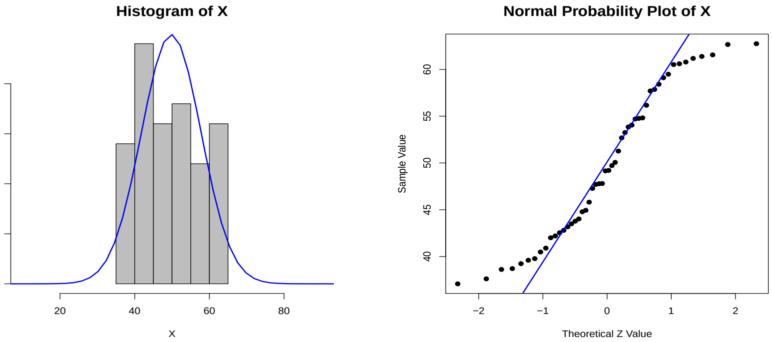


Figure: Histogram of non-normal, ”light tailed” data (left). Normal probability plot of the same data (right).

One-Factor ANOVA for Population Means $\mu_1, \mu_2, \ldots, \mu_I$

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  The populations might represent different ***groups*** or they might represent ***treatments*** in an experiment.

  We want to decide if there are any differences among the population means.

Example

A quality assurance study was carried out to compare **lead measurements** made in water sent to $I = 5$ **laboratories**.

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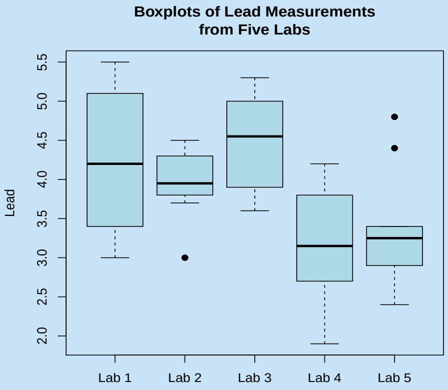
The **lead measurements** (μg/L) and their summary statistics are on the next slide.

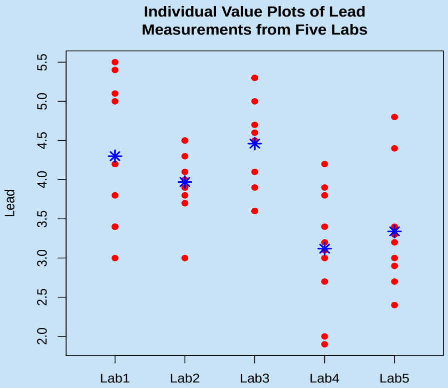
Measured Lead Concentrations

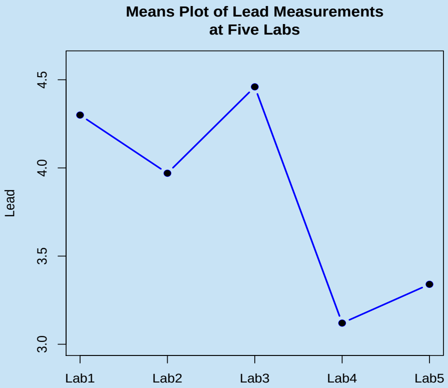
Lab 1	Lab 2	Lab 3	Lab 4	Lab 5
3.4	4.5	5.3	3.2	3.3
3.0	3.7	4.7	3.4	2.4
3.4	3.8	3.6	3.1	2.7
5.0	3.9	5.0	3.0	3.2
5.1	4.3	3.6	3.9	3.3
5.5	3.9	4.5	2.0	2.9
5.4	4.1	4.6	1.9	4.4
4.2	4.0	5.3	2.7	3.4
3.8	3.0	3.9	3.8	4.8
4.2	4.5	4.1	4.2	3.0

$\bar{X}_1 = 4.30$ $\bar{X}_2 = 3.97$ $\bar{X}_3 = 4.46$ $\bar{X}_4 = 3.12$ $\bar{X}_5 = 3.34$

$S_1 = 0.904$ $S_2 = 0.440$ $S_3 = 0.642$ $S_4 = 0.764$ $S_5 = 0.737$


Boxplots of Lead Measurements
from Five Labs
Lead
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
Lab 1
Lab 2
Lab 3
Lab 4
Lab 5


Individual Value Plots of Lead
Measurements from Five Labs
Lead
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
Lab1
Lab2
Lab3
Lab4
Lab5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Lead
3.0
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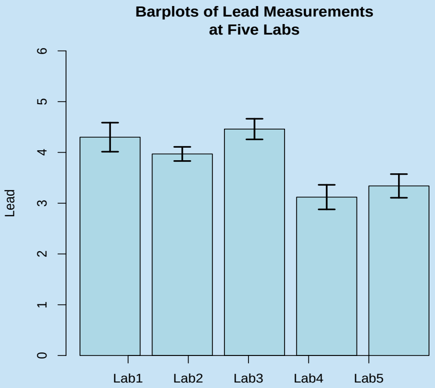
Barplots of Lead Measurements
at Five Labs
Lead
0
1
2
3
4
5
6
Lab1
Lab2
Lab3
Lab4
Lab5

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  The appropriate test is called the ***one-factor ANOVA F test***.

- **Comments**:
    - The sample sizes **don't** all have to be the same. But we'll only look at the equal-sample size case.

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  - The sample sizes **don't** all have to be the same. But we'll only look at the equal-sample size case.
  - The data can be **samples** from populations **or** responses to treatments in a **randomized experiment**.

- The **null hypothesis** is that there are no differences among the population means μ_1, μ_2, . . ., μ_I:

Null Hypothesis:

$$H_0 : \mu_1 \; = \; \mu_2 \; = \; \cdots \; = \; \mu_I$$

- The **alternative hypothesis** is that there's *at least one difference* among the set of means:

Alternative Hypothesis: The alternative hypothesis will be

$$H_a : \text{At least two of the } \mu_i\text{'s are different}$$

- **Notation**:

$$\begin{aligned}
I &= \text{The number of treatment groups} \\
J &= \text{The common sample size for the } I \text{ groups} \\
X_{ij} &= \text{The } j\text{th observation in the } i\text{th treatment group} \\
\bar{X}_{i\cdot} &= \text{The sample mean for the } i\text{th treatment group} \\
S_i &= \text{The sample standard deviation for the } i\text{th treatment group} \\
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Note:

$$\bar{X}_{\cdot\cdot} = \frac{1}{I}\sum_{i=1}^{I}\bar{X}_{i\cdot}$$

(when the sample sizes are all the same).

Sums of Squares and the ANOVA Partition

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  - **SST** is the ***total sum of squares***, defined as

$$\text{SST} = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{..})^2,$$

which measures the **total** variation in the X_{ij}'s.

- (cont'd):
  - **SSTr** is the ***treatment sum of squares***, defined as

$$\text{SSTr} = \sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 = J\sum_{i=1}^{I}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2,$$

    which measures variation **between** the treatment group means due to both **treatment effects** and **random error**.

- (cont'd):
  - **SSE** is the ***error sum of squares***, defined as

$$\text{SSE} = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{i\cdot})^2,$$

which measures variation of the X_{ij}'s **within** treatment groups due to **random error**.

Proposition

ANOVA Partition of the Total Variation: It can be shown that

$$\text{SST} = \text{SSTr} + \text{SSE}.$$

- The **ANOVA partition** holds because we can write:

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$$\sum_i \sum_j (X_{ij} - \bar{X}_{\cdot\cdot})^2 \;=\; \sum_i \sum_j (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 + \sum_i \sum_j (X_{ij} - \bar{X}_{i\cdot})^2 \,,$$

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which is the **ANOVA partition.**

Example

For the data on lead measurements at five labs, software gives

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$$\begin{array}{ccccc} 36.758 & = & 13.813 & + & 22.945 \\ \Uparrow & & \Uparrow & & \Uparrow \\ \text{Total} & & \text{Between} & & \text{Within} \\ \text{variation} & & \text{groups} & & \text{groups} \\ & & \text{variation} & & \text{variation} \end{array}$$

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Degrees of Freedom:

SST has $IJ - 1$ **df**

SSTr has $I - 1$ **df**

SSE has $I(J-1) = IJ - I$ **df**

- To see why:
  - The IJ deviations $X_{ij} - \bar{X}_{..}$ used to compute **SST** are subject to the **one constraint** that they **sum to zero**, i.e.

$$\sum_i \sum_j (X_{ij} - \bar{X}_{..}) = 0,$$

so only $\boldsymbol{IJ - 1}$ of them are "free to vary" (i.e. any $IJ - 1$ of them determines the remaining one).

- (cont'd):
  - The I deviations $\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}$ used to compute **SSTr** are subject to the **one constraint** that that they **sum to zero**, i.e.

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so only $\boldsymbol{I-1}$ of the deviations are "free to vary" (i.e. any $I-1$ of them determines the remaining one).

- (cont'd):
  - The IJ deviations $X_{ij} - \bar{X}_{i\cdot}$ used to compute **SSE** are subject to the I **constraints** that they **sum to zero *within* each of the I groups**, i.e.

$$\sum_j (X_{ij} - \bar{X}_{i\cdot}) = 0 \qquad \text{for each } i = 1, 2, \ldots, I$$

    Thus within each of the I samples, only $J - 1$ deviations are "free to vary" (i.e. any $J - 1$ of them determines the remaining one).

Additive Property of Degrees of Freedom:

$$\text{df for SST} \;=\; \text{df for SSTr} + \text{df for SSE}$$

since

$$IJ - 1 \;=\; (I-1) + I(J-1).$$

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  Example: A *sample variance* S^2 is a **mean square.**

- (cont'd)
  - The ***mean square for treatments***, denoted **MSTr**, is

$$\text{MSTr} = \frac{\text{SSTr}}{I-1}.$$

- (cont'd)
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(when the sample sizes are all the same).

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Thus **MSE** is the **average** (or **pooled**) **sample variance**.

- **MSTr** and **MSE** *are* directly comparable.

The One-Factor ANOVA F Test

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  It will be **large** when there are **differences** among $\mu_1, \mu_2, \ldots, \mu_I$.

Large values of F provide **evidence against H_0 in favor of H_a : At least two of the μ_i's are different.**

- Now suppose the I samples are from $N(\mu_1, \sigma)$, $N(\mu_2, \sigma)$ $\ldots$, $N(\mu_I, \sigma)$ distributions and that they were drawn *independently* of each other.

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  Alternatively, the samples could be from **non-normal** populations as long as the common sample size J is **large**.

Sampling Distribution of the Test Statistic Under H_0:

If F is the one-factor ANOVA F test statistic, then when

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I$$

is true,

$$F \sim F(I-1, I(J-1)).$$

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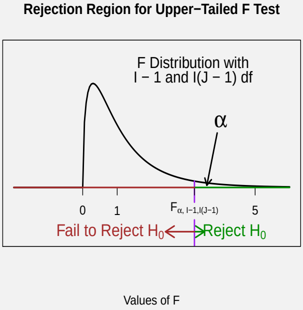
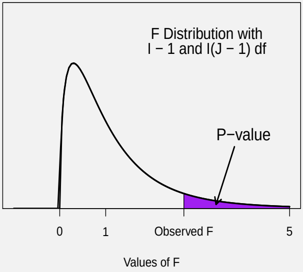
  - The ***p-value*** as the **tail area to the right of the observed F value.**

Rejection Region: The **rejection region** is the **set of F values** in the tail of the $F(I-1, I(J-1))$ curve to the **right of $F_{\alpha,\, I-1,\, I(J-1)}$**:



P-Value: The **p-value** is the **tail area** under the $F(I-1, I(J-1))$ curve to the **right** of the **observed** F:

P−Value for Upper−Tailed F Test



The ANOVA Table

- ANOVA results are summarized in an ***ANOVA table***:

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Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Treatment	$I-1$	SSTr	MSTr = SSTr/(I-1)	MSTr/MSE	p
Error	$I(J-1)$	SSE	MSE = SSE/(I(J-1))		
Total	IJ-1	SST			

Exercise

For lead measurements made at five labs, the **ANOVA table** is:

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Treatment	4	13.813	3.453	6.77	0.000
Error	45	22.945	0.510		
Total	49	36.758			

a) Verify that **df for SSTr =** $I - 1$, that **df for SSE =** $I(J - 1)$, and that **df for SST =** $IJ - 1$.

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c) Verify that the **mean squares** are the **sums of squares** divided by their **df**.

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b) Verify that **SST = SSTr + SSE** and that the **df for SST = df for SSTr + df for SSE**.

c) Verify that the **mean squares** are the **sums of squares** divided by their **df**.

d) Verify that the F **statistic** is **MSTr** divided by **MSE**.

e) State the **hypotheses.**

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g) If there are significant differences among the five labs, describe the nature of those differences (using the plots of the data given earlier in these slides).

- For comparing **two population means** μ_1 and μ_2, the **ANOVA** F **test** and a *two-sided* **pooled two-sample** t **test** are **equivalent.**

- For comparing **two population means** μ_1 and μ_2, the **ANOVA** F **test** and a *two-sided* **pooled two-sample** t **test** are **equivalent.**

  The **square** of the t **statistic** is the F **statistic**, and the **p-values** will be the **same.**

Example

An example in a previous set of slides presented results of a computer simulation to compare the time (in seconds) to complete a semiconductor manufacturing process using one and two operators.

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Here are the summary statistics:

One Operator			Two Operators		
m	$=$	16	n	$=$	16
$\bar{X}$	$=$	373.6	$\bar{Y}$	$=$	374.8
S_1	$=$	7.8	S_2	$=$	7.3

If we carry out a **(pooled) two-sample t test** of

$$\begin{aligned} H_0 : \mu_1 &= \mu_2 \\ H_a : \mu_1 &\neq \mu_2 \end{aligned}$$

we get:

Pooled t Test Statistic	**P-Value**
$t = -0.445$	**0.6596**

If we carry out a **one-factor ANOVA**, we get:

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Treatment	1	11.3	11.3	0.198	0.6596
Error	30	1710.2	57.0		
Total	31	1721.5			

If we carry out a **one-factor ANOVA**, we get:

Source of Variation	df	Sum of Squares	Mean Square	f	P-value
Treatment	1	11.3	11.3	0.198	0.6596
Error	30	1710.2	57.0		
Total	31	1721.5			

We see that $t^2 = F$ and the **p-values** for the two tests are the same.

- In general, the **square** of a t random variable is an F random variable.

Proposition

If

$$T \sim t(\nu)$$

then

$$T^2 \sim F(1, \nu).$$