

Statistical Methods

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Topics

- 1 Digression: Type I and II Errors and Their Probabilities (Cont'd)

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Objectives

Objectives:

- Compute the Type II error probability and power for the one-sample z test for μ .

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Digression: Type I and II Errors and Their Probabilities (Cont'd) (8.1, 8.2)

Type II Error Probabilities and the Power of a Test

- We denote the **probability** of making a **Type II error** (when H_a is true) by β :

$$\beta = P(\text{Type II Error}).$$

- The **power** of a test is the **probability** that you **don't** make a Type II error (when H_a is true):

$$\text{Power} = 1 - \beta.$$

(i.e. the **probability** that you **reject** H_0 when H_a is true).

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- Takeaway:

- The more **power** a test has, the more likely it is to detect **departures from H_0** .

Example: Clinical trial to test the effectiveness of a new drug, i.e.

H_0 : The drug has **no effect**

H_a : The drug has **an effect**

The more **power** the hypothesis test has, the more likely it is to detect **an effect** (if there is one).

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Type II Error Probability and Power for the One-Sample z Test

- The **power** is (relatively) easy to compute for the **one-sample z test**, which is used (instead of the t test) when the **population standard deviation σ is known**.

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One-Sample z Test Statistic:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

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Sampling Distribution of the Test Statistic Under H_0 :

If Z is the one-sample z test statistic, then when

$$H_0 : \mu = \mu_0$$

is true,

$$Z \sim N(0, 1).$$

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- The $N(0, 1)$ curve gives us:
 - The **rejection region** as the **extreme 100 α % of z values** (in the direction(s) specified by H_a).
 - The **p -value** as the **tail area(s) beyond the observed z value** (in the direction(s) specified by H_a).

Example

Suppose X_1, X_2, \dots, X_n are a random sample from a $N(\mu, \sigma)$ distribution, where σ is known, and we want to test

$$H_0 : \mu = \mu_0$$

$$H_a : \mu > \mu_0$$

The **test statistic** is

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

The **decision rule** (rejection region approach) is

Reject H_0 if $Z > z_\alpha$
 Fail to reject H_0 if $Z \leq z_\alpha$

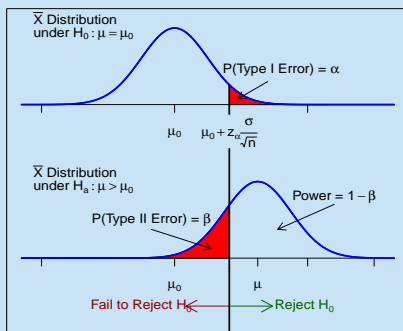
or equivalently

Reject H_0 if $\bar{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$
 Fail to reject H_0 if $\bar{X} \leq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$

where z_α is the 100(1 - α)th percentile of the $N(0, 1)$ distribution.

The **Type II error probability β** is the *shaded area* under the bottom curve on the next slide. The **power** is the *unshaded area*.

Type I and II Error Probabilities and Power for the One-Sample Z Test



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To compute β and the **power**, first recall that

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right),$$

and so

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

When $H_a : \mu > \mu_0$ is true,

$$\begin{aligned} \beta &= P(\text{Type II error}) = P(\text{Not rejecting } H_0) \\ &= P\left(\bar{X} \leq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\right) \\ &= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) \\ &= P\left(Z \leq z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) \\ &= \Phi\left(z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) \quad (1) \end{aligned}$$

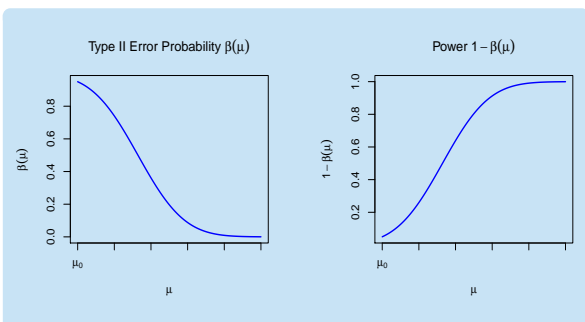
where $\Phi(z) = P(Z \leq z)$ denotes the **cdf** of the $N(0, 1)$ distribution.

Of course the **power** of the test is

$$\text{Power} = 1 - \beta.$$

β depends on the value of μ , so it's more appropriate to denote the **Type II error probability** as $\beta(\mu)$ and the **power** as

$$\text{Power} = 1 - \beta(\mu).$$



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- For lower-tailed and two-tailed z tests, expressions for $\beta(\mu)$ are found in a similar manner.

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Digression: Type I and II Errors and Their Probabilities (Cont'd)

Alternative Hypothesis	Type II Error Probability $\beta(\mu)$	
$H_a : \mu > \mu_0$	$\Phi\left(z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$	(1)
$H_a : \mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$	(2)
$H_a : \mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$	(3)

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Digression: Type I and II Errors and Their Probabilities (Cont'd)

Exercise

Suppose we have a sample of size $n = 16$ and we want to test

$$H_0 : \mu = 10$$

$$H_a : \mu > 10$$

using a level of significance $\alpha = 0.05$. Suppose also that we know that $\sigma = 4$, so the z test is appropriate.

- a) Find $\beta(12)$, the **probability of a Type II error** when $\mu = 12$.

Hint: $z_{0.05} = 1.64$, and you should get $\beta(12) = 0.3594$.

- b) Find the **power** of the test when $\mu = 12$.

Hint: You should get $\beta(12) = 0.6406$.

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Digression: Type I and II Errors and Their Probabilities (Cont'd)

- c) Which of the following would you expect to be true?

$$\beta(12) < \beta(13) < \beta(14)$$

or

$$\beta(12) > \beta(13) > \beta(14)$$

Hint: Refer to the graphs of $\beta(\mu)$ and $1 - \beta(\mu)$ on the next slide.

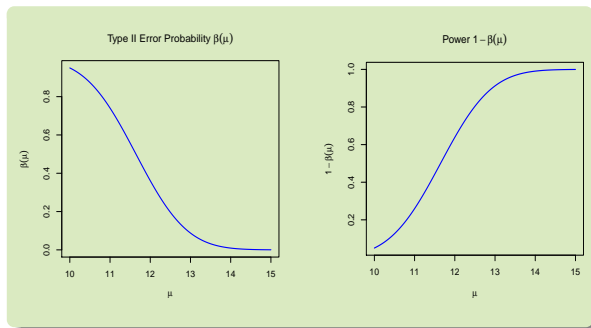
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Digression: Type I and II Errors and Their Probabilities (Cont'd)

- For t tests (and other tests), expressions for $\beta(\mu)$ are more complicated.

(See the textbook.)

The **Type II error probability** and **power** can be obtained using software.

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Digression: Type I and II Errors and Their Probabilities (Cont'd)

The Sample Size Needed to Attain a Desired Type II Error Probability for the One-Sample z Test

- For **fixed** μ , as n **increases**, $\beta(\mu)$ **decreases**, and therefore the **power increases** (see (1), (2), and (3)).
A **larger sample size** results in a **more powerful** test.
- We can determine **how big** n **needs to be** in order to attain a **desired** $\beta(\mu)$ (for any specified value of μ).
(See the textbook or the **optional** sides ahead for details).

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Digression: Type I and II Errors and Their Probabilities (Cont'd)

(Optional Slide)

Example

Consider again a test of

$$H_0 : \mu = 10$$

$$H_a : \mu > 10$$

using a level of significance $\alpha = 0.05$, and that $\sigma = 4$.

Now suppose we want n to be **large enough** so that the **Type II error probability** when $\mu = 12$ is **0.20**, which will give a **power of 0.80**.

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(Optional Slide)

We need n to be **large enough** that

$$\beta(12) = 0.20.$$

From (1), this means we need

$$0.20 = \Phi\left(z_\alpha + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) = \Phi\left(1.64 + \frac{10 - 12}{4/\sqrt{n}}\right).$$

But from a $N(0, 1)$ table,

$$0.20 = \Phi(-0.84)$$

(i.e. $z_{0.20} = 0.84$). This implies

$$1.64 + \frac{10 - 12}{4/\sqrt{n}} = -0.84.$$

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(Optional Slide)

Solving for n gives

$$n = \left(\frac{4(1.64 + 0.84)}{10 - 12}\right)^2 = 24.60, \quad (5)$$

so we'd need a **sample size of at least $n = 25$** .

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(Optional Slide)

- Generalizing from (5):

Sample Size for a Desired Type II Error Probability:

The sample size n for which a level α test has a desired Type II error probability β at the specified value μ is

$$n = \begin{cases} \left(\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu}\right)^2 & \text{for a one-tailed (upper or lower) test} \\ \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu}\right)^2 & \text{for a two-tailed test} \end{cases} \quad (6)$$

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(Optional Slide)

Exercise

Consider again a test of

$$H_0 : \mu = 10$$

$$H_a : \mu > 10$$

using $\alpha = 0.05$, and suppose again that $\sigma = 4$.

- a) Use (6) to determine how big n needs to be for the **Type II error probability** to be **0.10** when $\mu = 11$.

Hint: $z_{0.10} = 1.28$ and $z_{0.05} = 1.64$.

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(Optional Slide)

- b) Use (6) to determine how big n needs to be for the **Type II error probability** to be **0.10** when $\mu = 12$.
- c) If we want the **Type II error probability** to be **0.10** when $\mu = 13$, will n need to be **larger** or **smaller** than the one in Part *b*?
- d) If we want the **Type II error probability** to be **0.20** when $\mu = 12$, will n need to be **larger** or **smaller** than the one in Part *b*?

(Optional Slide)

- For ***t* tests** (and **other tests**), expressions for the **sample size n** needed to attain a desired **Type II error probability** or **power** are more complicated.

(See the textbook.)

The **sample size n** can be obtained using software.

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