Statistical Methods

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Topics

Objectives

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- State the main properties of F distributions.
- Carry out an F test for two population standard deviations σ_1 and σ_2.

F Distributions

- Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are random samples from a $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$ distributions, respectively, and that they were drawn *independently* of each other. Then the random variable

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \tag{1}$$

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$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \tag{1}$$

follows an ***F distribution*** with $\boldsymbol{m-1}$ ***numerator df*** and $\boldsymbol{n-1}$ ***denominator df***, denoted $\boldsymbol{F(m-1, n-1)}$.

- Also, the **reciprocal** of $\boldsymbol{F}$ follows an $\boldsymbol{F(n-1, m-1)}$ distribution (the **df** get **swapped**), i.e.

$$\frac{1}{F} = \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F(n-1, m-1).$$

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    2. The **numerator** and **denominator df** control the shape, center, and spread of the distribution.

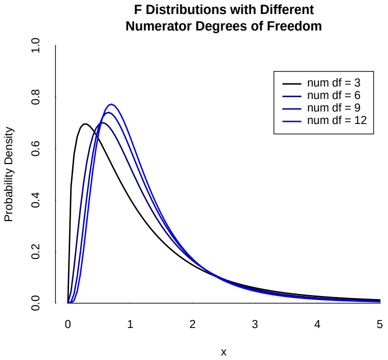

Figure: $F(m-1, 10)$ distributions with different values of m (left); $F(6, n-1)$ distributions with different values of n (right).

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Proposition

Suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are a random samples from *any* distributions whose standard deviations are σ_1 and σ_2, respectively, and that the samples were drawn *independently* of each other. Then **if m and n are large**,

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(m-1, n-1)$$

approximately.

F Test for Two Population Standard Deviations σ_1 and σ_2

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- We'll see how to use the samples to decide if the **population standard deviations** σ_1 and σ_2 are different.

  The appropriate test is called the ***F test for σ_1 and σ_2***.

- The **null hypothesis** is that no difference between the population standard deviations σ_1 and σ_2:

Null Hypothesis:

$$H_0 : \sigma_1 = \sigma_2$$

- The **alternative hypothesis** will depend on what we're trying to "prove":

Alternative Hypothesis: The alternative hypothesis will be one of

1. $H_a : \sigma_1 > \sigma_2$ (**one-sided, upper-tailed**)
2. $H_a : \sigma_1 < \sigma_2$ (**one-sided, lower-tailed**)
3. $H_a : \sigma_1 \neq \sigma_2$ (**two-sided, two-tailed**)

depending on what we're trying to verify using the data.

F Test Statistic for σ_1 and σ_2:

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$$F = \frac{S_1^2}{S_2^2}$$

- S_1 and S_2 are estimators of $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$, respectively, so ...
  1. F will be approximately **one** (most likely) if $\sigma_1 = \sigma_2$.
  2. It **larger than one** (most likely) if $\sigma_1 > \sigma_2$.
  3. It will be **smaller than one** (most likely) if $\sigma_1 < \sigma_2$.

1. **Large** values of F provide **evidence against** H_0 **in favor of** $H_a : \sigma_1 > \sigma_2$.
2. **Small** values of F provide **evidence against** H_0 **in favor of** $H_a : \sigma_1 < \sigma_2$.
3. **Large** ***and*** **small** values of F provide **evidence against** H_0 **in favor of** $H_a : \sigma_1 \neq \sigma_2$.

Sampling Distribution of the Test Statistic Under H_0:

If $F = S_1^2/S_2^2$ is the F test statistic, then when

$$H_0 : \sigma_1 = \sigma_2$$

is true,

$$F \;\sim\; F(m-1, n-1).$$

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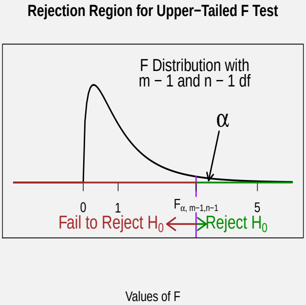
  - The ***rejection region*** as the **extreme 100α% of F values** (in the direction(s) specified by H_a).
  - The ***p-value*** as the **tail area(s) beyond the observed F value** (in the direction(s) specified by H_a).

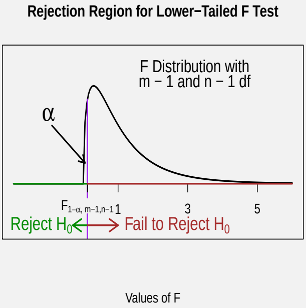
- We'll use $F_{\alpha,m-1,n-1}$ to denote the F **critical value** that's the $100(1-\alpha)$th percentile of the $F(m-1,n-1)$ distribution (i.e. the it has probability α to its **right**).

Rejection Region: The **rejection region** is the **set of F values** in the tail of the $F(m-1, n-1)$ curve:

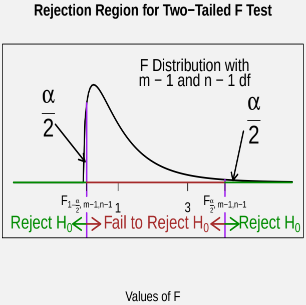
1. To the **right of $F_{\alpha,\, m\,-\,1,\, n\,-\,1}$** if the alternative hypothesis is $H_a : \sigma_1 > \sigma_2$:



2. To the **left of** $F_{1-\alpha, m-1, n-1}$ if the alternative hypothesis is $H_a : \sigma_1 < \sigma_2$:

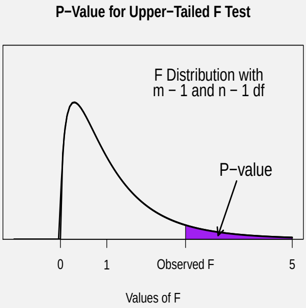


3. To the **left of** $F_{1-\alpha/2, m-1, n-1}$ **and right of** $F_{\alpha/2, m-1, n-1}$ if the alternative hypothesis is $H_a : \sigma_1 \neq \sigma_2$:

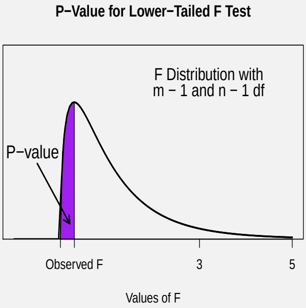


P-Value: The **p-value** is the **tail area** under the $F(m-1, n-1)$ curve:

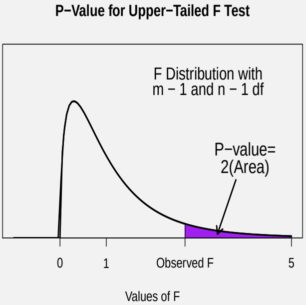
1. To the **right** of the **observed** F if the alternative hypothesis is $H_a : \sigma_1 > \sigma_2$:

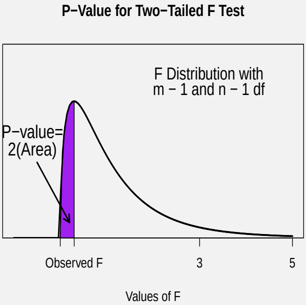


2. To the **left** of the **observed** F if the alternative hypothesis is $H_a : \sigma_1 < \sigma_2$:



3. To the **left of F or right of F**, whichever is **smaller**, then **multiplied by two** if the alternative hypothesis is $H_a : \sigma_1 \neq \sigma_2$:





Exercise

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To determine whether calcium affects **blood pressure**, twenty one men were randomized two groups.

The first consisted of $m = 10$ men given a **calcium** supplement for 12 weeks, and the second of $n = 11$ men given a **placebo** that appeared identical to the actual supplement.

The variable of interest is the **decrease** in blood pressure, as shown below.

Calcium Group (X)			Placebo Group (Y)		
Begin	End	Decrease	Begin	End	Decrease
107	100	7	123	124	-1
110	114	-4	109	97	12
123	105	18	12	113	-1
129	112	17	102	105	-3
112	115	-3	98	95	3
111	116	-5	114	119	-5
107	106	1	119	114	5
112	102	10	114	112	2
136	125	11	110	121	-11
102	104	-2	117	118	-1
			130	133	-3

The summary statistics are:

Calcium	Placebo
$m = 10$	$n = 11$
$\bar{X} = 5.00$	$\bar{Y} = -0.27$
$S_1 = 8.7$	$S_2 = 5.9$

Carry out an F **test** to decide if the (unknown) **population standard deviation** σ_1 for people who take **calcium** is **greater than** the **population standard deviation** σ_2 for people who take a **placebo**. Use level of significance $\alpha = 0.05$.

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Hints: You should get $F = 2.17$ and a critical value $F_{0.05,9,10} = 3.02$.

- **Comment**: If $F \sim F(m-1, n-1)$,

$$\alpha = P\left(F > F_{\alpha,m-1,n-1}\right) = P\left(\frac{1}{F} < \frac{1}{F_{\alpha,m-1,n-1}}\right). \quad (2)$$

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But $1/F \sim F(n-1, m-1)$ (df swapped), so (2) implies that

$$F_{1-\alpha,n-1,m-1} = \frac{1}{F_{\alpha,m-1,n-1}}.$$

- **Comment**: If $F \sim F(m-1, n-1)$,

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$$F_{1-\alpha,n-1,m-1} \;=\; \frac{1}{F_{\alpha,m-1,n-1}}.$$

This eliminates the need for F distribution tables to show critical values in *both* tails of the distribution.

Exercise

Below are summary statistics for two independent samples from **normal** populations.

Sample	Sample Size	Sample Standard Deviation
1	10	1.76
2	16	3.05

a) Carry out a **one-sided**, **lower tailed** test of

$$\begin{aligned} H_0 : \sigma_1 &= \sigma_2 \\ H_a : \sigma_1 &< \sigma_2 \end{aligned}$$

where $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_1}$ are the **first** and **second population standard deviations**, respectively. Use level of significance $\boldsymbol{\alpha = 0.05}$.

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where σ_1 and σ_1 are the **first** and **second population standard deviations**, respectively. Use level of significance $\alpha = 0.05$.

Hints: You should get $F = 0.333$ and the **critical value** $F_{0.95,9,15} = 0.332$.

b) Carry out a **two-sided** test of

$$H_0 : \sigma_1 = \sigma_2$$
$$H_a : \sigma_1 \neq \sigma_2$$

Use level of significance $\alpha = 0.10$.

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Use level of significance $\alpha = 0.10$.

Hints: You should get $F = 0.333$ and the **critical values** $F_{0.95,9,15} = 0.332$ and $F_{0.05,9,15} = 2.59$.