

# Introduction to Statistics

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September 22, 2019

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## Topics

1 Continuous Random Variables and the Normal Distribution

2 Finding Areas Under Normal Curves

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## Objectives

### Objectives:

- Interpret the probability density curve of a continuous random variable.
- Recognize a normal distribution, and interpret its mean and standard deviation.
- Recognize the standard normal distribution.
- Find areas under normal curves using Table II.

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## Continuous Random Variables and the Normal Distribution (6.1, 6.2, 6.3)

### Probability Density Curves

- If  $X$  is a **continuous** random variable, its **probability distribution** is represented by a **density curve**:
  - The curve can be thought of as an "idealized", smooth version of the **histogram** of a **large population**.
  - The **area** under the curve **above any interval of  $x$  values** represents the **proportion** of individuals whose  $x$  values lie in that interval (i.e. it represents the **probability** that a randomly selected individual's  $x$  value will fall in that interval).

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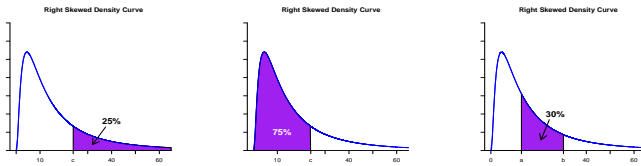
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- (density curves, cont'd)
  - Every *density curve* has the following properties:
    1. The entire curve lies on or above the  $x$ -axis.
    2. The **total area** under the curve **equals 1.0** (representing 100% of the population).

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The Normal Distribution

- By far the most widely applicable continuous probability distribution is the normal distribution, which is represented by the normal density curve.

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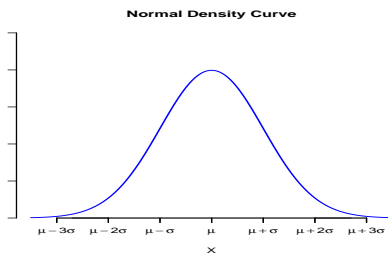


Figure: Normal density curve with mean  $\mu$  and standard deviation  $\sigma$ .

- There are actually **many** normal curves, one for each pair of values for its **mean  $\mu$**  and **standard deviation  $\sigma$** .  
These **parameters** are interpreted as the **population mean** and **population standard deviation**.
- $\mu$  is the **center** of the distribution, and  $\sigma$  measures its **spread**.

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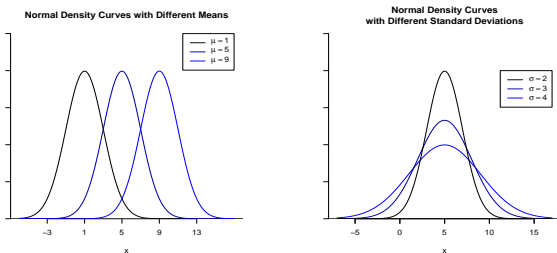


Figure: Normal distributions with different values of  $\mu$ , but the same  $\sigma$  (left), and with the same  $\mu$ , but different values of  $\sigma$  (right).

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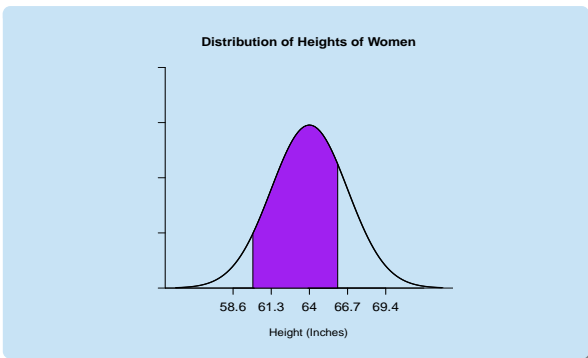
**Example**

Heights of women aged 20 to 29 follow a **normal distribution** with **mean  $\mu = 64$  inches** and **standard deviation  $\sigma = 2.7$  inches**.

This normal curve is shown on the next slide.

The **shaded area** corresponds to the **proportion** of women who are **between 60 and 66 inches** tall.

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### The Empirical Rule (or 68-95-99.7 Rule) Revisited

- **The Empirical Rule (or 68-95-99.7 Rule):** If a variable  $x$  in a population follows a **normal distribution** with mean  $\mu$  and standard deviation  $\sigma$ , then
  1. **68%** of the population lies within **one  $\sigma$**  of  $\mu$ .
  2. **95%** of the population lies within **two  $\sigma$ 's** of  $\mu$ .
  3. **99.7%** of the population lies within **three  $\sigma$ 's** of  $\mu$ .

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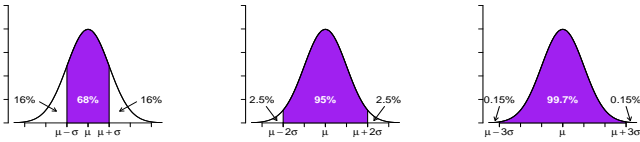
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### The Standard Normal Curve

- The normal distribution with  $\mu = 0$  and  $\sigma = 1$  is called the **standard normal distribution**.  
It's shown on the next slide.

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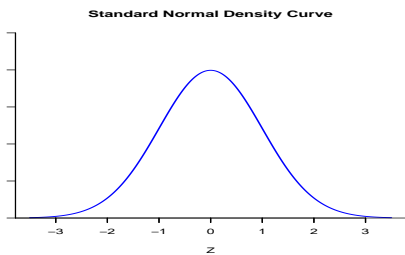


Figure: Standard normal density curve ( $\mu = 0$  and  $\sigma = 1$ ).

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# Finding Areas Under Normal Curves (6.1, 6.2, 6.3)

- Recall that the **z-score** for an individual's  $x$  value is:

$$z = \frac{x - \mu}{\sigma}$$

- In order to find **areas** under **any normal curve**, we'll need the facts on the next slide.

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If a variable  $x$  follows a **normal distribution** with **mean  $\mu$**  and **standard deviation  $\sigma$** , then

- The **standardized** version of that variable,

$$z = \frac{x - \mu}{\sigma}$$

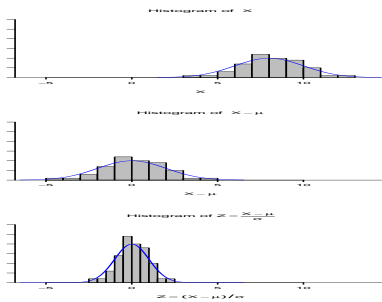
follows a **standard normal** distribution.

- The **proportion** of individuals whose  $x$  values lie between two numbers  $a$  and  $b$  is the **same** as the **proportion** whose  $z$ -scores lie between

$$z = \frac{a - \mu}{\sigma} \quad \text{and} \quad z = \frac{b - \mu}{\sigma}$$

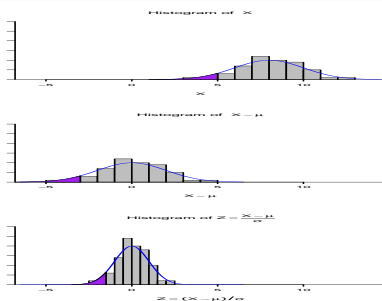
and this proportion is the **area** between these two  $z$  values under the **standard normal curve**, which can be obtained from **Table II**.

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**Figure:** Histogram of a variable  $x$  following a normal distribution with  $\mu = 8$  and  $\sigma = 2$  (top); histogram after subtracting  $\mu = 8$  from each  $x$  value (center); histogram of  $z$ , the standardized version of  $x$  (bottom). The bottom plot is a standard normal distribution.

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**Figure:** Histogram of a variable  $x$  following a normal distribution with  $\mu = 8$  and  $\sigma = 2$  (top); histogram after subtracting  $\mu = 8$  from each  $x$  value (center); histogram of  $z$ , the standardized version of  $x$  (bottom). The shaded areas in the three plots are the same.

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- Thus to find the **area** under **any normal curve**, use the following.

**Finding Areas Under a Normal Curve:** To find the area between two values  $a$  and  $b$  under a normal curve whose mean and standard deviation are  $\mu$  and  $\sigma$ :

1. **Sketch the normal curve** associated with the variable  $x$ .
2. **Shade the area** of interest and mark its delimiting  $x$  value(s).

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(cont'd):

3. **Find the  $z$ -score(s)** for the delimiting  $x$  values of Step 2. For example, if the shaded area of Step 2 is delimited by two values  $a$  and  $b$ , convert them to  $z$ -scores using:

$$z = \frac{a - \mu}{\sigma} \quad \text{and} \quad z = \frac{b - \mu}{\sigma}$$

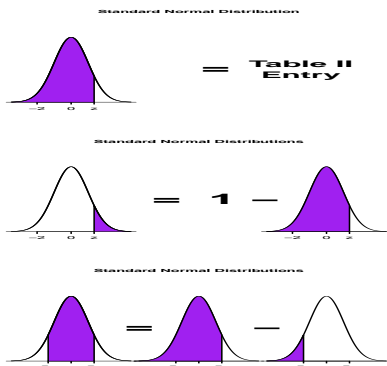
4. **Refer to Table II** to find the **area** under the **standard normal curve** delimited by the  $z$ -score(s) of Step 3.

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- **Table II** only gives areas to the **left** of a  $z$ -score.

The figures below and the exercise ahead show how to find the area to the **right** of a  $z$ -score and **between two**  $z$ -scores.

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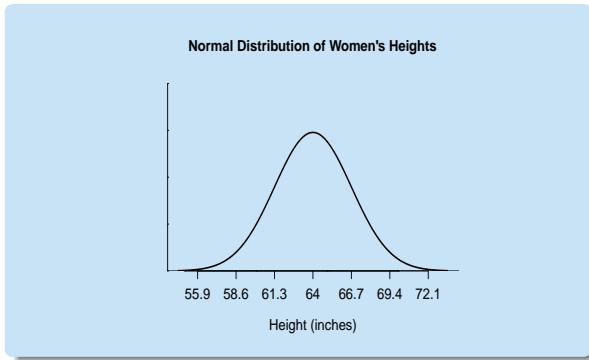
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Example

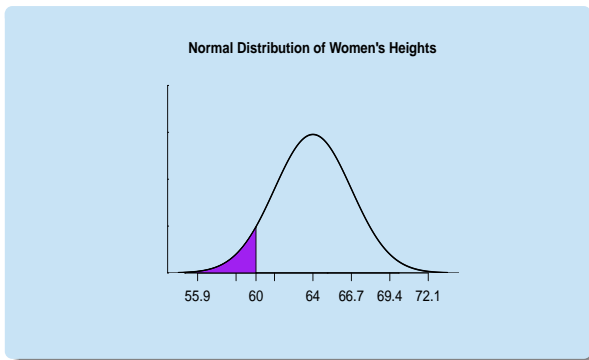
Recall that **heights of women aged 20 - 29 follow a normal distribution with mean  $\mu = 64$  inches and standard deviation  $\sigma = 2.7$  inches.**

This is the **normal density curve** on the next slide.



First we'll find the **percentage** of women that are **shorter** than **60** inches tall.

This is the **shaded area** under the density curves on the next slides.



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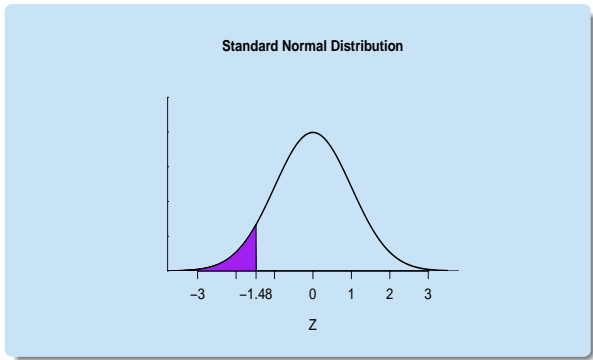
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The **z-score** for a 60 inch tall woman is

$$z = \frac{60 - 64}{2.7} = -1.48,$$

so the **percentage** of women **shorter** than 60 inches tall is the same as the **percentage** whose **z-scores** are **below -1.48**.

From **Table II**, a **proportion 0.0694** is **shorter** than 60 inches, so the **percentage** is **6.94%**.

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Now we'll find the **percentage** of women that are **taller** than 60 inches.

This is the **shaded area** under the density curves on the next slides.

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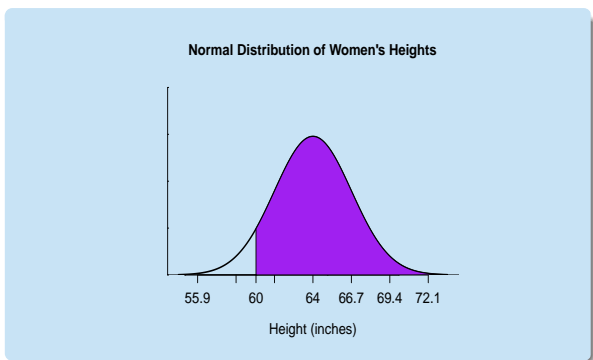
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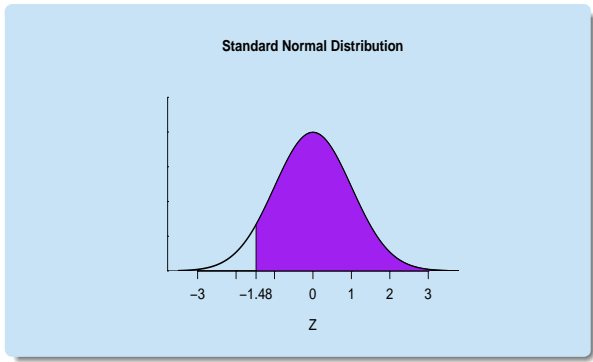
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Because the *total area* under a normal curve is **1.0** (corresponding to **100%** of women's heights), and a *proportion* **0.0694** is *shorter* than 60 inches (found previously), the *proportion taller* than 60 inches is

$$1 - 0.0694 = 0.9306.$$

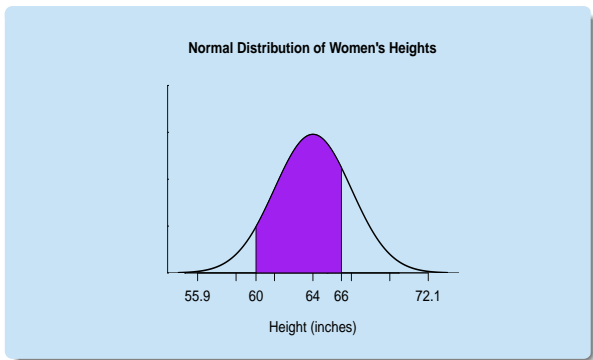
Thus the the **percentage** of women **taller** than **60** inches is **93.06%**.

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Now we'll find the **percentage** of women that are **between 60** and **66** inches tall.

This is the **shaded area** under the density curves on the next slides.

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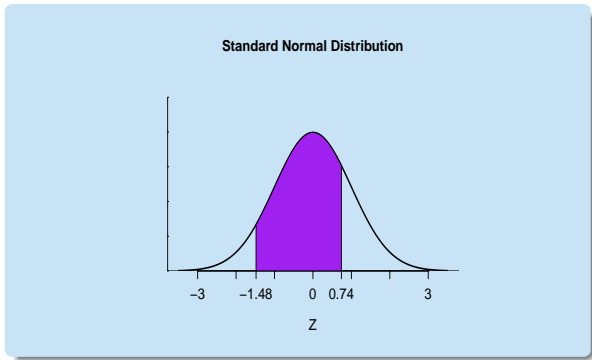
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The **z-score** for a 66 inch tall woman is

$$z = \frac{66 - 64}{2.7} = 0.74,$$

so the **proportion** of women **shorter** than 66 inches tall is the same as the **proportion** whose **z-scores** are **below 0.74**.

From **Table II**, this **proportion** is **0.7704**.

Thus, because a **proportion 0.0694** is **shorter** than 60 inches (found previously), the **proportion between 60 and 66** inches is

$$0.7704 - 0.0694 = 0.7010.$$

Therefore the **percentage between 60 and 66** inches is **70.1%**.

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**Exercise**

Refer to the previous example.

- a) **Sketch the density curve and shade the area** corresponding to the **percentage** of women **shorter** than **62** inches.
- b) Find the **percentage** of women that are **shorter** than **62** inches.
- c) **Shade the area** under the density curve corresponding to the **percentage** of women that are **taller** than **67** inches.
- d) Find the **percentage** of women that are **taller** than **67** inches.

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- e) **Shade the area** corresponding to the **percentage** of women that are **between 62 and 67** inches tall.

- f) Find the **percentage** of women that are **between 62 and 67** inches tall.

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