Introduction to Statistics

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Topics

Objectives

Objectives:

- Interpret conditional probabilities.
- Obtain conditional probabilities from contingency tables.
- (Optional) Use the Conditional Probability Rule to compute conditional probabilities.
- Use the General Multiplication Rule to compute probabilities.

Conditional Probability and the General Multiplication Rule (4.5)

Introduction

- Recall that two events are ***dependent*** when **the occurrence of one impacts whether or not the other will occur.**

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- A ***conditional probability*** is the probability of one event occurring under the condition that another one has occurred.

- We denote a **conditional probability** by $\boldsymbol{P(B \mid A)}$, that is,

$$P(B \mid A) \quad = \quad \text{The conditional probability of } B \text{ occurring, } \textit{given} \text{ that } A \text{ has occurred.}$$

The symbol "|" is read as "given".

Example

Joe and Tom are each dealt a card face down from a standard **52**-card deck. Let

$$\begin{aligned} A &= \text{The event that Joe has an Ace} \\ B &= \text{The event that Tom has an Ace} \end{aligned}$$

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Because there are **four** aces in the deck,

$$P(A) = \frac{\mathbf{4}}{\mathbf{52}}.$$

Now suppose Joe turns his card over and it's an ace. That leaves only **three** aces in the remaining **51** cards, so the **conditional probability** that Tom has an ace, **given** that Joe has an ace, is **3/51**, which we write as

$$P(B \mid A) = \frac{\mathbf{3}}{\mathbf{51}}.$$

Exercise

A jar contains **4** red jelly beans, **3** yellow ones, and **3** green ones. You'll close your eyes and grab a jelly bean. Then you'll grab another one **without having put the first one back in the jar**. Let

A = The event that the **first** jelly bean is **yellow**

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a) Find $P(A)$, the probability that the first jelly bean is yellow.

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a) Find $P(A)$, the probability that the first jelly bean is yellow.

b) Find $P(B \mid A)$, the **conditional probability** that the second jelly bean will be yellow, **given** that the first one was yellow.

Finding Conditional Probabilities from Contingency Tables

- The next example shows how to find conditional probabilities from contingency tables.

Example

Here's the contingency table showing the frequencies (counts), by gender and student level, at a large university:

		Student Level			
		Undergraduate	Professional	Graduate	Total
Gender	Male	18,208	249	4,436	22,893
	Female	12,643	651	**2,660**	**15,954**
	Total	30,851	900	7,096	38,847

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	Total	30,851	900	7,096	38,847

Consider randomly selecting one of the 38,847 individuals from this university. Let

A = The event that the selected individual is a **female**

B = The event that the selected individual is a **graduate student**

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Then because **2,660** of the **15,954** females at the university are graduate students, the **conditional probability** that the student is a graduate student, ***given*** that she's female, is

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We interpret this as saying that **16.6%** ***of the females*** at the university are graduate students.

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- The last example illustrates the following general method for finding conditional probabilities from contingency tables on the next slide.

Conditional Probabilities from Contingency Tables: Consider randomly selecting an individual from a population described by a contingency table. Let

R = The event that the selected individual falls in a given **row**

C = The event that the selected individual falls in a given **column**

Then

$$P(\text{C} \mid \text{R}) = \frac{\text{Cell Frequency}}{\text{Row Marginal Total Frequency}}$$

and

$$P(\text{R} \mid \text{C}) = \frac{\text{Cell Frequency}}{\text{Column Marginal Total Frequency}}$$

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a) Find the **conditional** probability that a student is an **undergraduate**, **given** that he's **male.**

b) Interpret the probability in Part *a* as a **percentage.**

c) Find the **conditional** probability that a student is **male**, **given** that the student is an **undergraduate**.

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Refer to the contingency table in the last example.

a) Find the **conditional** probability that a student is an **undergraduate**, **given** that he's **male**.

b) Interpret the probability in Part *a* as a **percentage**.

c) Find the **conditional** probability that a student is **male**, **given** that the student is an **undergraduate**.

d) Interpret the probability in Part *c* as a **percentage**.

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This says that **when A and B are *independent*, the occurrence of A has *no impact* on whether B will occur.**

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Consider tossing a coin twice. Let

A = The event that the coin land H on the **first** toss

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A = The event that the coin land H on the **first** toss

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Then

$$P(A) = \mathbf{0.5} \qquad \text{and} \qquad P(B) = \mathbf{0.5},$$

and since outcomes of coin tosses are **independent**, getting H on first toss has **no impact** on the outcome of the second toss, so

$$P(B \mid A) = P(B) = \mathbf{0.5}.$$

(Optional Section) The Conditional Probability Rule

- The ***Conditional Probability Rule*** shows how we can calculate conditional probabilities:

Conditional Probability Rule: The *conditional probability* of an event B, *given* that another event A has occurred, can be calculated by

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}$$

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Consider randomly selecting one of the 38,847 individuals from this university. Let

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From the table,

$$P(A) = \frac{\mathbf{15,954}}{\mathbf{38,847}}$$

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Suppose we're told that the selected individual is a female. By the **Conditional Probability Rule**, the **conditional probability** that the student is a **graduate student**, given that she's **female**, is

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Suppose we're told that the selected individual is a female. By the **Conditional Probability Rule**, the **conditional probability** that the student is a **graduate student**, given that she's **female**, is

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)} = \frac{2,660/38,847}{15,954/38,847} = \frac{2,660}{15,954} = \mathbf{0.166}.$$

Note that this is the same answer as the one we got in an earlier example.

The General Multiplication Rule

- The ***General Multiplication Rule*** shows how to find $P(A \;\&\; B)$ when A and B **aren't** independent:

General Multiplication Rule: For **any** two events A and B (**not necessarily independent**):

$$P(A \;\&\; B) \;=\; P(A) \times P(B \mid A).$$

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If we think of A and B happening one after the other, this rule says that in order for A and B to **both** occur, A needs to occur (which happens with probability $P(A)$), **and then**, given that A occurred, B needs to occur (which happens with probability $P(B \mid A)$.

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We know from an earlier example that

$$P(A) = \frac{\mathbf{4}}{\mathbf{52}} \quad \text{and} \quad P(B \mid A) = \frac{\mathbf{3}}{\mathbf{51}}.$$

Exercise

Joe and **Tom** are each dealt a card from a standard **52**-card deck. Let

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We know from an earlier example that

$$P(A) = \frac{\mathbf{4}}{\mathbf{52}} \quad \text{and} \quad P(B \mid A) = \frac{\mathbf{3}}{\mathbf{51}}.$$

Use the General Multiplication Rule to find the probability that **both** players have aces.

Exercise

Suppose again a jar has **4** red jelly beans, **3** yellow ones, and **3** green ones. You'll grab a jelly bean, then grab another one **without having put the first one back in.** Let

A = The event that the **first** jelly bean is **yellow**

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Exercise

Suppose again a jar has **4** red jelly beans, **3** yellow ones, and **3** green ones. You'll grab a jelly bean, then grab another one **without having put the first one back in.** Let

A = The event that the **first** jelly bean is **yellow**
B = The event that the **second** jelly bean is **yellow**

We know from an earlier exercise that

$$P(A) = \frac{\mathbf{3}}{\mathbf{10}} \quad \text{and} \quad P(B \mid A) = \frac{\mathbf{2}}{\mathbf{9}}.$$

a) Use the **General Multiplication Rule** to find $P(A \;\&\; B)$, the probability that **both** jelly beans will be **yellow.**

b) Now find the probability that the **first** jelly bean will be **yellow** **and** the **second** one will be **red**.