Statistical Methods

Nels Grevstad

Metropolitan State University of Denver

ngrevsta@msudenver.edu

October 28, 2019

Notes

Topics

Notes

Objectives

Objectives:

- Carry out Tukey's multiple comparison procedure after a two-factor ANOVA with $K > 1$, and interpret the results.
- Interpret residuals and fitted values.
- Use residuals to check normality and constant standard deviation assumptions.

Notes

Multiple Comparisons for Two-Factor ANOVA ($K > 1$) when the Interaction Effect *Isn't* Significant

- When the no-interaction hypothesis H_{0AB} **is not** rejected and at least one of the two main effect hypotheses H_{0A} and H_{0B} **is** rejected, we can use **Tukey's procedure** to decide **which** levels of a factor differ.

Notes

Tukey's Multiple Comparison Procedure: If the two-factor ANOVA F test fails to reject H_{0AB}, but rejects H_{0A} or H_{0B}:

1. Choose an **overall familywise confidence level** $100(1-\alpha)\%$ (usually $\alpha = 0.05$ for a 95% confidence level).

Notes

2. For **Factor A comparisons**, compute the $I(I-1)/2$ **confidence intervals**:

$$\bar{X}_{i\cdot\cdot} - \bar{X}_{i'\cdot\cdot} \;\pm\; Q_{\alpha,I,IJ(K-1)}\sqrt{\frac{MSE}{JK}}\,.$$

   For **Factor B comparisons**, compute the $J(J-1)/2$ **confidence intervals**:

$$\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdot j'\cdot} \;\pm\; Q_{\alpha,J,IJ(K-1))}\sqrt{\frac{MSE}{IK}}\,.$$

3. For any interval that **doesn't contain zero**, deem those levels of the given factor to be **different**.

Notes

Multiple Comparisons for Two-Factor ANOVA ($K > 1$) when the Interaction Effect *Is* Significant

- When the no-interaction hypothesis H_{0AB} **is** rejected, we can use **Tukey's procedure** to decide **which *group* means** differ.

  This can be done by carrying out a ***one-factor* ANOVA** on the IJ ***groups*** following by the (*one-factor*) **Tukey procedure**.

Notes

Estimating Parameters (when $K > 1$)

- Recall (Slides 15) that the *full* **two-factor ANOVA model** is:

$$X_{ijk} \;=\; \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}\,.$$

Notes

Model Parameter Estimators: We estimate the unknown model parameters μ, α_i, β_j, and σ using the **estimators** $\boldsymbol{\hat{\mu}}$**,** $\boldsymbol{\hat{\alpha}_i}$**,** $\boldsymbol{\hat{\beta}_j}$, and $\boldsymbol{\hat{\sigma}}$ defined as:

Model Parameter	Estimator
μ	$\hat{\mu} \ = \ \bar{X}_{...}$
$\alpha_i \ = \ \mu_{i\cdot} - \mu$	$\hat{\alpha}_i \ = \ \bar{X}_{i..} - \bar{X}_{...}$
$\beta_j \ = \ \mu_{\cdot j} - \mu$	$\hat{\alpha}_i \ = \ \bar{X}_{.j.} - \bar{X}_{...}$
$\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$	$\hat{\gamma}_{ij} \ = \ \bar{X}_{ij.} - (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j)$
	$= \ \bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...}$
σ	$\hat{\sigma} \ = \ \sqrt{\text{MSE}}$

Notes

Fitted Values and Residuals

- The ***fitted value*** (or ***predicted value***) for the kth individual in the i,jth cell, $\boldsymbol{\hat{X}_{ijk}}$, is

$$\begin{aligned} \hat{X}_{ij} \ &= \ \hat{\mu} \ + \ \hat{\alpha}_i \ + \ \hat{\beta}_j \ + \ \hat{\gamma}_{ij} \\ &= \ \bar{X}_{...} \ + \ (\bar{X}_{i..} - \bar{X}_{...}) \ + \ (\bar{X}_{.j.} - \bar{X}_{...}) \\ & \quad + (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} \ + \ \bar{X}_{...}) \\ &= \ \bar{X}_{ij.} \ . \end{aligned}$$

  $\hat{X}_{ijk}$ is the value we'd predict, based on the data, for the response of the kth individual in the i,jth cell.

  It's just the $\boldsymbol{i,j}$**th group mean** $\boldsymbol{\bar{X}_{ij.}}$ (which is also the estimate of the true group mean μ_{ij}).

Notes

- The ***residual*** for the kth observation in the i,jth cell, $\boldsymbol{e_{ijk}}$, is defined as

$$\begin{aligned} e_{ijk} \ &= \ X_{ijk} - \hat{X}_{ijk} \\ &= \ X_{ijk} - (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij}) \\ &= \ X_{ijk} - \bar{X}_{ij.} \end{aligned}$$

  The **residual** $\boldsymbol{e_{ijk}}$ corresponds to the **random error** term $\boldsymbol{\epsilon_{ijk}}$ in the model.

  Note that a **residual** is just the **deviation** of an observed response $\boldsymbol{X_{ijk}}$ **away from** the **group mean** $\boldsymbol{\bar{X}_{ij.}}$.

Notes

- **Comment**: The **error sum of squares** (Slides 15) is the **sum of squared residuals**, i.e.

$$\text{SSE} \ = \ \sum_i \sum_j \sum_k e_{ijk}^2 \ .$$

Notes

Checking the Model Assumptions

- For the **ANOVA F test**, we assume the ϵ_{ijk}'s are iid $N(0, \sigma)$.

  Note that $\boldsymbol{\sigma}$ is assumed to be **constant** from one group to the next.

- **Checking the Normality Assumption**: Use a **histogram** or **normal probability plot** of the **residuals**.

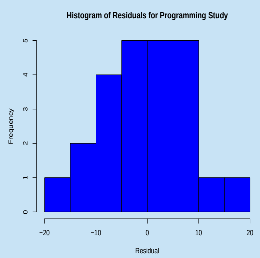
- **Checking the Constant σ Assumption**: Plot the **residuals** versus the **fitted values**.

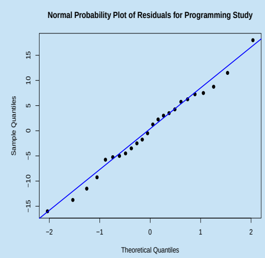
  Usually, when σ *isn't* constant, it increases with the group mean.

Notes

Example

For the study of factors affecting programmers' errors in predicting project completion times, a **histogram** and **normal probability plot** of the **residuals** are below.



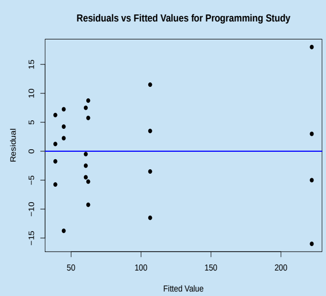


Notes

The **normality assumption** of the **errors** $\boldsymbol{\epsilon_{ijk}}$ appears to be met.

A plot of the **residuals** versus **fitted values** is on the next slide.

Notes



Notes

The amount of spread is roughly the same from group to group, so the **constant standard deviation assumption** appears to be met.

Thus the **ANOVA F test** results are valid.

Notes

Notes

Notes

Notes