

Introduction to Statistics

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Topics

- 1 Determining the Required Sample Size for Estimating μ
- 2 Confidence Interval for μ when σ is Unknown

Objectives

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- Determine the sample size required for the margin of error in a CI for a population mean to be no bigger than some specified value.
- Distinguish between the t distribution and the standard normal distribution.
- Compute and interpret a CI for a population mean when the population standard deviation isn't known.

Determining the Required Sample Size for Estimating

μ (8.2)

- Recall that the **margin of error** in a **one-mean z CI** for μ is

$$\text{Margin of Error} = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

Determining the Required Sample Size for Estimating

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- Recall that the **margin of error** in a **one-mean z CI** for μ is

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We can make the **margin of error as small as we want** by using a **large enough n** .

- Suppose we want the **margin of error** to **no bigger than** some value E .

Solving

$$z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq E$$

for n gives the **required sample size** (see the next slide).

Sample Size for Estimating μ : The sample size required for the margin of error in a CI for μ to be no bigger than some value E is

$$n \geq \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2,$$

which we **round up** to the nearest integer.

In practice, we usually don't know the value of σ , so we plug in a guess for its value (e.g. based on prior studies).

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Our goal is to **estimate** the true (unknown) **mean** amount of time μ (in minutes) using a **99% CI**.

A similar study in Seattle suggests that a reasonable guess for the Denver population standard deviation σ is **50** minutes.

How large should our sample size be if we want the **margin of error** to be **no bigger than 10** minutes?

We have

$$E = 10$$

$$z_{\alpha/2} = z_{0.005} = 2.58$$

$$\sigma = 50$$

so we'd need

$$n \geq \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2 = \left(\frac{2.58(50)}{10} \right)^2 = 166.41,$$

which we **round up** to **167**.

Exercise

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Exercise

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Suppose now, though, that we only need the **margin of error** to be **no bigger than 15** minutes.

- a) Will the **sample size** required for a **15-minute margin of error** be **larger** or **smaller** than the one required for a **10-minute margin of error**?

b) Calculate the **sample size** required for a **15-minute margin of error**. Compare it to the one required for a **10-minute margin of error**.

Confidence Interval for μ when σ is Unknown (8.3)

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Instead, we **estimate** σ by the sample standard deviation s , and then use the ***one-mean t CI*** described ahead.

Introduction to the t Distribution

- For the ***one-mean t CI***, we'll need a new probability distribution, the **t distribution**.

- Recall that the *standardized version* of the sample mean,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}},$$

follows a **standard normal** distribution (when the sample is from a *normal* population or n is *large*).

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- If we use the **estimate s in place of σ** , the *standardized version* of \bar{x} ,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}},$$

instead follows a so-called ***t distribution*** with $n - 1$ ***degrees of freedom*** (or ***df***).

- There's a different t **distribution** for each value of the **degrees of freedom**, $n - 1$.

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Several t distribution curves are shown on the next slide.

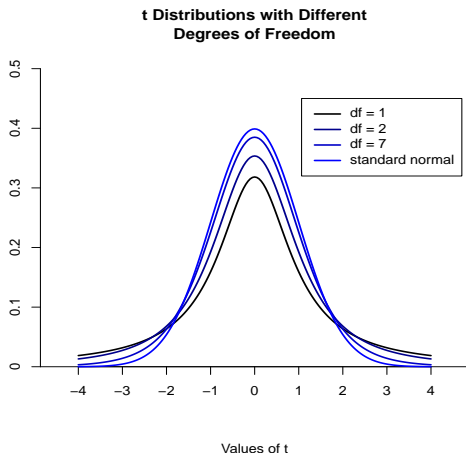


Figure: t distributions with different df along with the standard normal (z) curve.

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- As the **df increases**, the **t curves** get closer and closer to the **standard normal z curve**.

When the **df** is larger than about **40**, the **t** and **z curves** are practically **indistinguishable**.

The One-Mean t Confidence Interval for μ

- The next slide gives the procedure for computing a CI for μ when σ *isn't* known.

One-Mean t CI for μ : When the population standard deviation σ **is unknown**, the **one-mean t confidence interval** for the **unknown population mean μ** is

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}},$$

with **margin of error**

$$\text{Margin of Error} = t_{\alpha/2} \frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is a **t critical value**, obtained from the t distribution with $n - 1$ df, and α is either **0.10**, **0.05**, or **0.01**, depending on the level of confidence (see the next slides).

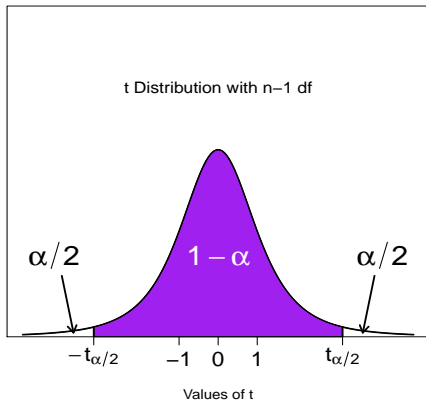
- $\alpha = 0.10$ for a **90%** level of confidence (so $1 - \alpha = 0.90$, $\alpha/2 = 0.05$, and the critical value is $t_{0.05}$)
- $\alpha = 0.05$ for a **95%** level of confidence (so $1 - \alpha = 0.95$, $\alpha/2 = 0.025$, and the critical value is $t_{0.025}$)
- $\alpha = 0.01$ for a **99%** level of confidence (so $1 - \alpha = 0.99$, $\alpha/2 = 0.005$, and the critical value is $t_{0.005}$)

(The actual values of $t_{0.05}$, $t_{0.025}$, and $t_{0.005}$ will depend on the df.)

(The CI and margin of error are valid when either the sample is from a **normal** population or the sample size n is **large**.)

- The *t critical value*, $t_{\alpha/2}$, is the t value for which the area to its **right** under the t distribution with $n - 1$ degrees of freedom is $\alpha/2$.

Depiction of $t_{\alpha/2}$



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The **sample mean** and **sample standard deviation** of the **11** observations are

$$\bar{x} = 48.4 \quad \text{and} \quad s = 18.1$$

The **population mean** μ and **standard deviation** σ are both **unknown**.

One goal of the study was to **estimate** the **true (unknown) population mean** insect-detection-distance μ using a **CI**.

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where the t **critical value**, $t_{0.025} = 2.228$, was obtained from **Table IV** using $n - 1 = 10$ **df**.

We can be **95% confident** that the true (unknown) mean μ is in this interval somewhere.

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Answer: Yes because 55 is contained in the interval.

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Answer: Yes because 55 is contained in the interval.

Is it plausible that μ is **as large as 65** cm?

Answer: No because 65 isn't contained in the interval.

Note (two slides back) that the **margin of error** in the estimate, $\bar{x} = 48.4$ cm, is **12.2** cm.

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More precisely, it tells us that the **sampling error is not likely** to be **larger** than **12.2** cm.

If we use a **99% confidence level**, the **t critical value** (from **Table IV** using $n - 1 = 10$ df) is $t_{0.005} = 3.169$, and the **CI** for μ is

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Note that the **99% CI** is **wider** than the **95% CI**.

Exercise

A highway safety researcher studying the design of a highway sign is interested in the **population mean distance** μ at which drivers are first able to read the sign.

A sample of $n = 16$ drivers reported that they first read the sign at the following distances (in ft):

440	490	600	540	540	600	380	440
360	510	490	400	490	540	440	490

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$$\bar{x} = 484.4 \quad s = 71.3$$

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- a) **Compute and interpret a 95% CI for the true (unknown) population mean** sign-readability-distance μ .

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- a) **Compute** and **interpret** a **95% CI** for the **true (unknown) population mean** sign-readability-distance μ .
- b) **Is it plausible**, based on the CI, that μ is **as large as 510 ft**? Explain.

The **sample mean** and **sample standard deviation** of the **16** observations are

$$\bar{x} = 484.4 \quad s = 71.3$$

- Compute** and **interpret** a **95% CI** for the **true (unknown) population mean** sign-readability-distance μ .
- Is it plausible**, based on the CI, that μ is **as large as 510 ft**? Explain.
- How big is the **margin of error** in the estimate, $\bar{x} = 484.4$, of μ ? How do you **interpret** this value?

d) Now compute a **99% CI** for μ .

- d) Now compute a **99% CI** for μ .
- e) Which CI is **wider**, the **95%** interval or the **99%** interval?