

Introduction to Statistics

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Paired Samples Test for μ_1 and μ_2

Topics

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Paired Samples Test for μ_1 and μ_2

Objectives

Objectives:

- Distinguish between paired samples and independent samples.
- Carry out a paired-samples t test for two population means μ_1 and μ_2 .

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Paired Samples Test for μ_1 and μ_2

Paired Samples Test for μ_1 and μ_2 (10.5)

Paired Samples

- **Paired samples** arise when **two samples** are collected in **pairs**, whereby each individual from the first population is **matched** with one in the second population, and a random sample of **pairs** is selected.

Matching of individuals is done in such a way that **within each pair**, the two individuals have **similar characteristics**.

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Exercise

Decide which of the following are **paired samples** and which are **independent samples**.

- To compare prices at two grocery stores, a random sample of food items is selected from one store, and the **same** items selected from the other store.
- To compare lengths of comedy and drama movies, a random sample of comedies is selected and a **separate** random sample of dramas is selected.

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Paired Samples Test for μ_1 and μ_2

- **Pairs samples** can also arise when **two measurements** are made **on each individual** in a study, one measurement under one condition and the other under a different condition.

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Paired Samples Test for μ_1 and μ_2

Example

To decide if college students retain information better by reading the material in a **book** or by watching it on a **DVD**, a sample of $n = 7$ students was each assigned one 19th-century novel to read and a different one to watch.

Afterward, they were given a 100-point **quiz** on each novel.

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Paired Samples Test for μ_1 and μ_2

These **two measurements** (quiz scores) for **each student** resulted in the following **paired samples**:

Student	Book Quiz Result	DVD Quiz Result
1	90	85
2	80	72
3	90	80
4	75	80
5	80	70
6	90	75
7	84	80

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Paired Differences

- The **pairwise differences** are obtained by subtracting each value in the second sample from its paired value in the first sample.

This is illustrated in the next example.

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Paired Samples Test for μ_1 and μ_2

Example

For the study to decide if college students retain information better by reading a book or by watching a DVD, the seven **paired differences** are shown in the right column below:

Student	Book Test Result	DVD Test Result	Difference
1	90	85	5
2	80	72	8
3	90	80	10
4	75	80	-5
5	80	70	10
6	90	75	15
7	84	80	4

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Paired Samples Test for μ_1 and μ_2

- When the samples are **paired**, we **shouldn't** use the two-mean t test for μ_1 and μ_2 (because that test is only appropriate for *independent* samples).

Instead, use the **paired samples t test** described ahead.

(The test can also be used to compare responses to **two treatments** in a **paired experiment**.)

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Paired Samples Test for μ_1 and μ_2

- The test considers the **paired differences** to be a **single random sample** from a (hypothetical) "**population of differences**".

We'll use the sample of *differences* to decide if the population means μ_1 and μ_2 are different.

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- The **null hypothesis** is:

Null Hypothesis:

$$H_0 : \mu_1 = \mu_2$$

- The **alternative hypothesis** will depend on what we're trying to **substantiate**:

Alternative Hypothesis: The alternative hypothesis will be one of

- $H_a : \mu_1 > \mu_2$ **(one-sided, upper-tailed test)**
- $H_a : \mu_1 < \mu_2$ **(one-sided, lower-tailed test)**
- $H_a : \mu_1 \neq \mu_2$ **(two-sided, two-tailed test)**

depending on what we're trying to substantiate in our study.

- The **test statistic** for the **paired samples t test** is

Paired t Test Statistic:

$$t = \frac{\bar{d} - 0}{s_d / \sqrt{n}}$$

where

\bar{d} = The sample mean of the n *differences*

and

s_d = The sample standard deviation of the n *differences*

When

$$H_0 : \mu_1 = \mu_2$$

is true, the **sampling distribution** of the test statistic t is a t distribution with $n - 1$ df.

(The **paired samples t test** is valid if either the sample of *differences* is from a **normal** population or the sample size n is **large** ($n \geq 30$).)

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- **Intuition:** It can be shown that the **mean** of the "**population of differences**", denoted μ_d , is equal to the difference between the two population means, i.e.

$$\mu_d = \mu_1 - \mu_2.$$

Thus the **paired samples t test statistic** can be viewed as a **one-mean t test statistic** for

$$H_0 : \mu_d = 0$$

using the **sample of differences**.

- t measures (approximately) how many **standard errors \bar{d}** is **away from** the claimed value **zero**.

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Paired Samples Test for μ_1 and μ_2

1. Values of t **close to zero** provide almost **no evidence against the null hypothesis**

$$H_0 : \mu_1 = \mu_2.$$

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Paired Samples Test for μ_1 and μ_2

2. **Positive** values of t provide **evidence against the null hypothesis in favor of**

$$H_a : \mu_1 > \mu_2.$$

3. **Negative** values of t provide **evidence against the null hypothesis in favor of**

$$H_a : \mu_1 < \mu_2.$$

4. **Positive and negative** values of t provide **evidence against the null hypothesis in favor of**

$$H_a : \mu \neq \mu_0.$$

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Paired Samples Test for μ_1 and μ_2

- The **p -value** is the probability that just by chance (under H_0) we'd get a test statistic value as far from zero, in the direction predicted by H_a , as the observed value.

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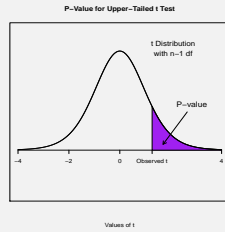
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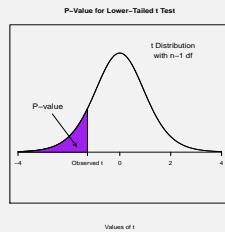
1. **P-value** = Area to the **right** of the observed t if the alternative hypothesis is $H_a : \mu > \mu_0$.



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Paired Samples Test for μ_1 and μ_2

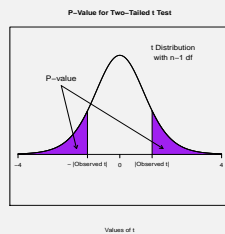
2. **P-value** = Area to the **left** of the observed t if the alternative hypothesis is $H_a : \mu < \mu_0$.



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Paired Samples Test for μ_1 and μ_2

3. **P-value** = Area to the **left** of $-|t|$ and **right** of $|t|$ if the alternative hypothesis is $H_a : \mu \neq \mu_0$.



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Paired Samples Test for μ_1 and μ_2

- As always, after choosing a **level of significance** α , we **reject H_0** if **p-value** $\leq \alpha$, otherwise fail reject H_0 .

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Example

A company wanted to know if attending a course on "how to be a successful salesperson" can increase the average sales of its employees.

The company sent $n = 6$ of its salespersons to attend this course.

The table on the next slide gives the one-week sales of these salespersons **before** and **after** they attended this course, along with the **differences** (after minus before).

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Paired Samples Test for μ_1 and μ_2

Salesperson	Sales After the Course	Sales Before the Course	Difference
1	18	12	6
2	24	18	6
3	24	25	-1
4	14	9	5
5	19	14	5
6	20	16	-4

The summary statistics for the **paired differences** shown above are

$$\bar{d} = 4.17 \quad \text{and} \quad s_d = 2.64$$

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Paired Samples Test for μ_1 and μ_2

We'll carry out a **paired-samples t test** to **decide if the sales increase** as a result of attending the course.

The **hypotheses** are

$$H_0 : \mu_1 = \mu_2$$

$$H_a : \mu_1 > \mu_2$$

where μ_1 is the true (unknown) population mean sales **after** attending the course, and μ_2 is the true mean **before**.

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Paired Samples Test for μ_1 and μ_2

The observed **test statistic** is

$$\begin{aligned} t &= \frac{\bar{d} - 0}{s_d / \sqrt{n}} \\ &= \frac{4.17 - 0}{2.64 / \sqrt{6}} \\ &= \mathbf{3.87}. \end{aligned}$$

Thus the **sample mean difference** in sales, $\bar{d} = 4.17$, is about **3.87 standard errors above zero**.

The **p-value** is the **probability** that we'd get a t value this far above zero (in either direction) if the **course had no effect** on sales.

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From the **upper tail** of the t distribution with $n - 1 = 5$ df, to the **right** of **3.87**,

$$\text{p-value} = 0.006$$

(obtained from the **table showing areas to the right of t**).

Thus we'd get a result like the one we got **only 0.6%** of the time **if the course had no effect** on sales.

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Paired Samples Test for μ_1 and μ_2

Using a **level of significance** $\alpha = 0.05$, the **decision rule** is

Reject H_0 if p-value < 0.05 .

Fail to reject H_0 if p-value ≥ 0.05 .

Because $0.006 < 0.05$, we **reject H_0** .

There's **statistically significant evidence** that the course results in increased sales.

The observed differences in sales (for the sample of six employed who attended the course) *cannot* be explained by chance variation (sampling error).

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Paired Samples Test for μ_1 and μ_2

Exercise

Refer to the study to decide if college students retain information better by reading a book or by watching a DVD.

Using the $n = 7$ paired differences given in that example, the summary statistics for the **differences** are

$$\bar{d} = 6.71 \quad \text{and} \quad s_d = 6.32$$

Carry out the **paired-samples t test** to **decide if there's any difference** in a student's ability to retain information by reading a **book** and by watching a **DVD**. Use a level of significance $\alpha = 0.05$.

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