# 1 Partial $F$ Tests

- We can use the *general linear test* approach to decide if adding a predictor to a model is "useful" for explaining variation in $Y_1, Y_2, \ldots, Y_n$ that's not already explained by the predictors in the model.

## 1.1 Partial $F$ Test for a Single $\beta_k$

- To test whether a predictor $X_k$ should be added to a model that *already* includes the other $p-2$ predictors $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1}$, define the **full model**:

$$Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_{p-1} X_{p-1} + \epsilon_i$$

and the **reduced model**:

$$Y_i = \beta_0 + \beta_1 X_1 + \cdots + \beta_{k-1} X_{k-1} + \beta_{k+1} X_{k+1} + \cdots + \beta_{p-1} X_{p-1} + \epsilon_i$$

- Then

$$\text{SSE}(F) = \text{SSE}(X_1, X_2, \ldots, X_{p-1})$$

and

$$\text{SSE}(R) = \text{SSE}(X_1, X_2, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})\,.$$

Note that

$$\begin{aligned} \text{SSE}(R) - \text{SSE}(F) &= \text{SSE}(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1}) - \text{SSE}(X_1, \ldots, X_{p-1}) \\ &= \text{SSR}(X_k | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1}) \end{aligned}$$

- To test

$$\begin{aligned} H_0 : \beta_k &= 0 \\ H_a : \beta_k &\neq 0 \end{aligned} \qquad (1)$$

using the *general linear test* approach, the ***F* test statistic** can be written in terms of the partial sum of squares $\text{SSR}(X_k | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})$ as:

$$\begin{aligned} F &= \frac{(\text{SSE(R)} - \text{SSE(F)})/(df_R - df_F)}{\text{SSE(F)}/df_F} \\ &= \frac{\text{SSR}(X_k | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})/1}{\text{SSE}(X_1, \ldots, X_{p-1})/(n-p)} \\ &= \frac{\text{SSR}(X_k | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})}{\text{MSE}(X_1, \ldots, X_{p-1})} \end{aligned} \qquad (2)$$

where

$$\text{MSE}(X_1, \ldots, X_{p-1}) = \frac{\text{SSE}(X_1, \ldots, X_{p-1})}{n-p}\,.$$

- When $H_0$ is true, the test statistic (2) follows an $\boldsymbol{F(1, n-p)}$ **distribution**. The **p-value** is the tail area to the **right** of the observed $F$ value under this distribution.

- This *partial* $F$ *test* for $\beta_k$ and the $t$ *test* for $\beta_k$ (Class Notes 12) are equivalent:

> **Fact 1.1** The $F$ statistic (2) for testing the hypotheses (1) about $\beta_k$ is the square of the $t$ statistic $t = (b_k - 0)/s\{b_k\}$ for testing those hypotheses (Class Notes 12), i.e.
>
> $$F = t^2,$$
>
> and the p-values for the two tests will be the same.

This says that the $t$ statistics and p-values for each predictor $X_k$ reported by statistical software relate to the **marginal effect of $\boldsymbol{X_k}$ on the mean response, given that the other predictors are already in the model**, i.e. *controlling* for (holding constant) the other predictors.

### 1.2 Partial $\boldsymbol{F}$ Test for Multiple $\boldsymbol{\beta_k}$'s

- The *partial* $F$ *test* can also be used to test whether **two or more predictors** should be added to a model that *already* includes the other predictors.

For example, to test for the marginal effects of $X_3$ and $X_4$, given that $X_1$ and $X_2$ are already in the model, the test statistic is

$$F = \frac{\text{SSR}(X_3, X_4 | X_1, X_2)/2}{\text{MSE}(X_1, \ldots, X_{p-1})},$$

and the p-value is the right tail area under the $F(2, n-p)$ distribution.

## 2 Coefficient of Partial Determination

- The ***coefficient of partial determination*** (or ***partial*** $\boldsymbol{R^2}$), denoted $\boldsymbol{R^2_{X_k|X_1,\ldots,X_{k-1},X_{k+1},\ldots,X_{p-1}}}$, is defined as

> **Partial $\boldsymbol{R^2}$:**
>
> $$R^2_{X_k|X_1,\ldots,X_{k-1},X_{k+1},\ldots,X_{p-1}} = \frac{\text{SSR}(X_k|X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})}{\text{SSE}(X_1, \ldots, X_{k-1},\ X_{k+1}, \ldots, X_{p-1})}$$

It measures the **proportion** of **remaining** (unexplained) **variation** in $\boldsymbol{Y}$, after fitting a model containing $\boldsymbol{X_1, \ldots, X_{k-1},\ X_{k+1}, \ldots, X_{p-1}}$, that can be **explained by** $\boldsymbol{X_k}$.