Taylor Polynomials: The Lagrange Error Bound

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In order to understand the rôle played by the Lagrange remainder and the Lagrange error bound in the study of power series, let's carry the standard examination of the geometric series a little farther than is usually done. That standard examination usually ends with the observation that

$$\sum_{k=0}^{n} x^{k} = \frac{1 - x^{n+1}}{1 - x}, \tag{1}$$

and so the sums on the left side of (1) converge to f(x) = 1/(1-x) iff |x| < 1. Of course, this conclusion arises from the fact that the limit on the right-hand side exists iff |x| < 1. But (1) contains other information from which we can learn a great deal more about the convergence of the geometric series.

Let us begin by rewriting (1) as

$$\sum_{k=0}^{n} x^{k} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

This latter equation can be rewritten in two different ways:

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1-x}, \text{ and}$$
 (2)

$$\frac{1}{1-x} - \sum_{k=0}^{n} x^{k} = \frac{x^{n+1}}{1-x}.$$
(3)

Notice that equation (2) has the form

$$f(x) = \sum_{k=0}^{n} a_k x^k + R_n(x).$$
(4)

This is exactly the form that we see in Taylor's formula with Lagrange remainder, which we will state very soon. But let us defer thoughts connected with this observation for a while.

It is equation (3) that we want to deal with now. Let h be any number with 0 < h < 1. If $x \in [-h, h]$, then $x \le h < 1$, so $|1 - x| = 1 - x \ge 1 - h$. This guarantees that

$$\frac{1}{|1-x|} = \frac{1}{1-x} \le \frac{1}{1-h}.$$
(5)

Consequently, we may write, from (3) and (5),

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n} x^{k} \right| = \left| \frac{x^{n+1}}{1-x} \right|$$
(6)

$$\leq \frac{h^{n+1}}{1-h},\tag{7}$$

no matter what the number $x \in [-h, h]$ may be. Thus, on any interval of the form [-h, h], where 0 < h < 1, the values of the polynomial $\sum_{k=0}^{n} x^k$ approximate the values of the function $f: x \mapsto 1/(1-x)$ uniformly well.

This turns out to be an extremely important piece of information. It has among its consequences the fact that if 0 < h < 1 the definite integrals of the polynomials $1 + x + \cdots + x^n$ on intervals $[x_1, x_2] \subseteq [-h, h]$ approximate the definite integrals of f well.

Now let's note that

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$
(8)

An analysis very similar to the one above, but of the polynomials

$$\frac{d}{dx}\sum_{k=0}^{n}x^{k} = \sum_{k=1}^{n}kx^{k-1},$$
(9)

yields—when 0 < h < 1—the inequality

$$\left|\frac{1}{(1-x)^2} - \sum_{k=1}^n kx^{k-1}\right| \le \frac{h^n[(1+h)n+1]}{(1-h)^2},\tag{10}$$

valid for every $x \in [-h, h]$. The fraction on the right side of (10) goes to zero as n grows without bound (use L'Hôpital's rule together with the fact that |h| < 1 to see this), and so the derivatives of the polynomials $\sum_{k=0}^{n} x^k$ approximate the derivative of the function f uniformly well on any interval of the form [-h, h], where 0 < h < 1. In other words, because of inequalities like (7) and (10) we can approximate the calculus of the function f by using the calculus of the polynomials $\sum_{k=0}^{n} x^{k}$ instead.

Let us now turn from this specific example to more general functions f. Taylor's formula with Lagrange remainder asserts that if f is any function defined on some neighborhood (-H, H), H > 0, of the origin and possessing derivatives of all orders up to and including n + 1 for some positive integer n, and if $x \neq 0$ is such that -H < x < H, there must be a number ξ , somewhere in the interval whose endpoints are 0 and x, such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} + \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}.$$
 (11)

The first thing we notice is that, as advertised earlier, equation (11) has the same form as equation (4). Thus, the Lagrange remainder (and, in fact, each of the other forms in which the remainder can be put) plays the same rôle for general functions that the expression $x^{n+1}/(1-x)$ plays for the function $x \mapsto 1/(1-x)$. We may rewrite (11) as

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} \right| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \right|,$$
(12)

and, if we know that the derivative $f^{(n+1)}$ is bounded by, say, M_n , on some interval of the form [-h, h], we obtain the Lagrange error bound for the Taylor polynomial of degree n:

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} \right| \leq \frac{M_{n}}{(n+1)!} |x^{n+1}|.$$
(13)

If we can show that such bounds M_n exist for all n and if we also can find the right kind of information about the way in which the numbers M_n grow as $n \to \infty$, we will be able to show that the Taylor polynomials for f approximate f uniformly well on intervals of the form [-h, h].

What is more, a similar inequality holds for the Taylor polynomials for f', and so those polynomials approximate f' uniformly well on intervals of the form [-h, h]. Thus, the calculus of the Taylor polynomials of f approximates the calculus of the original function f on such intervals. It turns out that we can acquire the necessary information for most of the functions that one encounters in an elementary calculus course—as long as we require that 0 < h < R, where R is the radius of convergence of the Taylor series we are looking at. It is not an exaggeration to say that this is the real reason that we study power series: *Power series allow us to approximate the calculus of the function* f by way of the calculus of the Taylor polynomials. And it is the Lagrange error bound that often plays a central rôle in establishing this fact for a given function f.