AP Calculus 2015 AB FRQ Solutions

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1 Problem 1

1.1 Part a

The rate of change, in cubic feet per hour, of the volume of water in the pipe at time t is $R(t) - D(t)$, where

$$R(t) = 20 \sin\left(\frac{t^2}{35}\right), \tag{1}$$

and

$$D(t) = -0.04t^3 + 0.4t^2 + 0.96t. \tag{2}$$

Thus,

$$\begin{aligned} v(t) &= R(t) - D(t) & (3)\\ &= 0.04t^3 - 0.4t^2 - 0.96t + 20 \sin\left(\frac{t^2}{35}\right). & (4) \end{aligned}$$

Integrating numerically, we find that the amount of water, in cubic feet, that flows into the tank in the eight-hour time interval $0 \le t \le 8$ is

$$\int_0^8 R(t)\, dt = \int_0^8 20 \sin\left(\frac{t^2}{35}\right)\, dt \sim 76.57035. \tag{5}$$

1.2 Part b

The rate of change of volume of water in the pipe at time $t = 3$ is

$$v(3) = -0.31363 \text{ cubic feet per hour.} \tag{6}$$

This is negative, and $v'(t)$ is continuous, so the volume of water in the tank is decreasing when t is near 3.

Note: We have phrased our answer this way because very few authors give a definition for the phrase "increasing at $x = a$ Instead, the usual definition is for "increasing on an interval".

1.3 Part c

From a plot, we see that $v(t)$ is zero at a value $t = t_0$ near $t = 3$, negative immediately to the left of this zero, and positive to the right. It follows from the First Derivative Test that this zero of $v(t)$ gives a relative minimum for the amount of water in the pipe. Neither endpoint can be a global minimum, because $v(t)$ is negative to the right of $t = 0$ and $v(t)$ is positive to the left of $t = 8$. Solving numerically, we find that $t_0 \sim 3.27155$ hours. Hence, volume is minimal at about $t = 3.27155$ hours.

1.4 Part d

There are intially 30 cubic feet of water in the pipe, and the pipe can hold 50 cubic feet of water before overflowing, so, using what we have seen in Part a, above, we can determine the time of overflow by solving the equation

$$30 + \int_0^t v(\tau)\, d\tau = 50. \tag{7}$$

for t.

Note: Solution of this equation is not required. However, numerical methods give $t \sim 8.23202$, which lies outside the domain we were given. We conclude that the pipe doesn't overflow during the specified interval.

2 Problem 2

2.1 Part a

First, we solve numerically for the solution x_0 of $f(x) = g(x)$ that lies near $x = 1$, obtaining, $x_0 \sim 1.03283$. Numerical integration gives the desired area:

$$\int_0^{x_0} [g(x) - f(x)]\, dx + \int_{x_0}^2 [f(x) - g(x)]\, dx \sim 2.00434. \tag{8}$$

Note: The integral is not elementary, and we have no choice but to integrate numerically.

2.2 Part b

The volume of the solid is, again by numerical integration,

$$\int_{x_0}^{2} [f(x) - g(x)]^2 \, dx \sim 1.8316. \tag{9}$$

Note: The integral is not elementary, and we have no choice but to integrate numerically.

2.3 Part c

We know that $h(x) = f(x) - g(x)$, so the rate at which $h(x)$ changes is

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(e^{x^2-2x} - x^4 + 6.5x^2 - 5x - 1\right) &(10)\\ &= 2(x-1)e^{x^2-2x} - 4x^3 + 13x - 5. &(11)\end{aligned}$$

When $x = 1.8$ we have

$$h'(1.8) \sim -3.81172, \tag{12}$$

and this is the rate that $h(x)$ changes with respect to x when $x = 1.8$.

3 Problem 3

3.1 Part a

Using data from the table, we find that the approximate value of the derivative $v'(16)$ is

$$v'(1.8) \sim \frac{v(20) - v(12)}{20 - 12} = \frac{240 - 200}{20 - 12} = 5 \text{ meters/min.} \tag{13}$$

3.2 Part b

The definite integral $\int_0^{40} |v(t)| \, dt$ gives the actual distance, in meters, that Johanna traveled in the time interval $0 \le t \le 40$. The right Riemann sum approximation of this distance is $12 \cdot |200| + 8 \cdot |240| + 4 \cdot |-220| + 16 \cdot |150| = 7600$ m.

3.3 Part c

If Bob's velocity at time t is $B(t) = t^3 - 6t^2 + 300$ meters per minute, then his acceleration at time t is $B'(t) = 3t^2 - 12t$. Thus his acceleration at time $t = 5$ is $v'(5) = 15$ meters per minute per minute.

3.4 Part d

Bob's average velocity over the interval $[0, 10]$ is

$$\frac{1}{10}\int_0^{10} B(\tau)\,d\tau = \frac{1}{10}\int_0^{10} \left(\tau^3 - 6\tau^2 + 300\right)\,d\tau \tag{14}$$

$$= \frac{1}{10}\left[\frac{\tau^4}{4} - 2\tau^3 + 300\tau\right]\Bigg|_0^{10} = 350 \text{ meters per minute.} \tag{15}$$

4 Problem 4

4.1 Part a

See Figure 1.

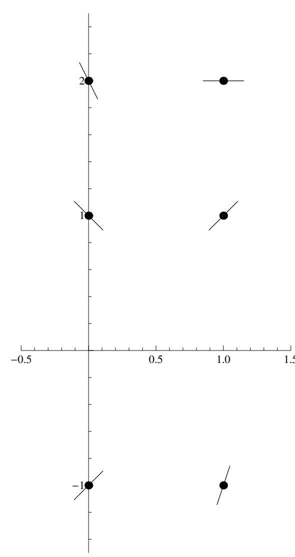

Figure 1: Problem 4, Part a: Slope Field

4.2 Part b

If $y' = 2x - y$, then $y'' = 2 - y' = 2 - 2x + y$. In the second quadrant, $x < 0$ and $y > 0$, so $y'' > 0$ throughout that quadrant. It follows that any solution curve that passes through any part of the second quadrant must be concave upward there.

4.3 Part c

If f is a solution of the differential equation $y' = 2x - y$ for which $f(2) = 3$, then $f'(2) = 2 \cdot 2 - 3 = 1$, so f doesn't have a critical point at $x = 2$. Thus, f has neither a maximum nor a minimum at $x = 2$.

4.4 Part d

If $y = mx + b$ is a solution of the differential equation $y' = 2x - y$, then, on the one hand, direct differentiation of the solution shows that we must have $y' = m$. On the other hand, the differential equation implies that we must also have $y' = 2x - (mx + b) = (2 - m)x - b$. Consequently, $m = (2 - m)x - b$, which we can rewrite and $(2 - m)x - (m + b) \equiv 0$. It follows that $m = 2$ and $b = -2$.

4.5 Remark

The differential equation $y' = 2x - y$ can be rewritten as $y' + y = 2x$, which has the form $y' + p(x)y = q(x)$. Such a differential equation can always be solved by choosing a function $P(x)$ such that $P'(x) = p(x)$ and multiplying the differential equation through by $e^{P(x)}$. After doing so, we find that the new equation can always be put in the form

$$\frac{d}{dx}\left[ye^{P(x)}\right] = q(x)e^{P(x)}. \tag{16}$$

All that then remains is to carry out the integration on the right side.

Applying this technique to the current equation, we obtain

$$ky'e^x + ye^x = 2xe^x, \text{ or;} \tag{17}$$

$$\frac{d}{dx}\left[ye^x\right] = 2xe^x \tag{18}$$

An integration by parts on the right side now gives

$$ye^x = 2xe^x - 2e^x + C \tag{19}$$

$$y = 2x - 2 + Ce^{-x}. \tag{20}$$

This is the general solution to the differential equation.

The initial value problem of Part c can now be solved by substituting 3 for y and 2 for x, to learn that we must take $C = e^2$. This gives the solution $y = 2x - 2 + e^{2-x}$. We can use this solution to confirm the conclusions obtained above for Part c of the problem.

We can also obtain the linear solution required in Part d by taking $C = 0$ in the general solution.

5 Problem 5

5.1 Part a

By the First Derivative Test, the function f has a relative maximum at $x = -2$, where $f'(x)$ changes sign from positive to negative. There are no other relative maxima.

5.2 Part b

The graph of f is both concave downward and decreasing on any open interval where f' is both decreasing and negative. The open intervals where f' displays such behavior are $(-2, -1)$ and $(1, 3)$. (And any open sub-interval of either of these two intervals.)

5.3 Part c

A point of inflection is to be found wherever the derivative changes its behavior from increasing to decreasing or from decreasing to increasing. These things happen at $x = -1$, at $x = 1$, and at $x = 3$, so these are the x-coordinates of points of inflection for f.

5.4 Part d

If $f(1) = 3$, then, by the Fundamental Theorem of Calculus,

$$f(x) = 3 + \int_1^x f'(\tau)\, d\tau. \tag{21}$$

Thus,

$$f(4) = 3 + \int_1^4 f'(\tau)\, d\tau = 3 - 12 = -9, \tag{22}$$

and

$$f(-2) = 3 + \int_1^{-2} f'(\tau)\, d\tau = 3 + 9 = 12. \tag{23}$$

6 Problem 6

6.1 Part a

At $(-1, 1)$, we have

$$y' = \frac{1}{3 \cdot (1)^2 - (-1)} = \frac{1}{4}, \tag{24}$$

so an equation for the line tangent to this curve at the point $(-1, 1)$ is

$$y = 1 + \frac{1}{4}(x + 1). \tag{25}$$

6.2 Part b

If the tangent to this curve is to be vertical, y' must not exist. This can be so only if $3y^2 - x = 0$, or $x = 3y^2$. (But see the remarks below.) Substituting this latter expression for x in the equation of the curve, we obtain $y^3 - (3y^2)y = 2$, or $y^3 = -1$. Thus, $y = -1$. But if $y = -1$ and $y^3 - xy = 2$, then $x = 3$. Thus, the only point on this curve where the tangent line can be vertical is the point with coordinates $(3, -1)$.

6.3 Part c

We differentiate the equation given for y' with respect to x:

$$\frac{d}{dx}y' = \frac{d}{dx}\left[\frac{y}{3y^2 - x}\right] \text{ or} \tag{26}$$

$$y'' = \frac{y'(3y^2 - x) - y(6yy' - 1)}{(3y^2 - x)^2}. \tag{27}$$

Setting $x = -1$ and $y = 1$, while taking $y = 1/4$ as found in Part a, above, we find that

$$y''\bigg|_{(-1,1)} = \frac{1}{32}. \tag{28}$$

6.4 Some Remarks

I've given here the solution for Part b of this problem as I expect that most successful examinees will answer it, and as I think the Exam Committee intended it to be answered. But this solution is devoid of logical support: the observation that the derivative is not defined at a point on a curve need not imply that the tangent line to the curve at that point is vertical.

In this case, the solution works. But it works because of things, including the Implicit Function Theorem, that students of AP Calculus don't know—and shouldn't be expected to know. Implicit differentiation with respect to x applied to an equation $F(x, y) = 0$, which is the unnamed source of the expression the problem gives for the derivative, is meaningless at a point where the partial derivative $F_2(x, y)$ vanishes. That's because the equation $F(x, y) = 0$ may fail to define, in any interval centered at such a point, a unique differentiable function φ of x that satisfies $F\big[x, \varphi(x)\big] = 0$. And if there is no such function, it's meaningless to talk about the derivative of that function.

That's exactly what happens at the point $(3, -1)$ in this problem, where

$$F(x, y) = y^3 - xy - 2. \tag{29}$$

There is no open interval centered at $x = 3$ on which there is a unique differentiable function φ such that $\varphi(3) = -1$ and $F[x, \varphi(x)] = 0$. In fact, if x is a number just smaller than 3, there aren't even any values of y near -1 that satisfy $F(x, y) = 0$. So the fact that the denominator in this derivative vanishes at $(3, -1)$ really signifies only the meaninglessness of the question "What's y' at $(3, -1)$?" and not what the examiners probably wanted it to signify.

In fact, the "solution" given above works—but not because of anything I've said yet, nor because of anything that examinees are likely to say. It works because the partial derivative $F_1(x, y)$ doesn't vanish at $(3, -1)$, while the partial derivative $F_2(x, y)$ does. From the fact that $F_1(3, -1) \neq 0$, we can conclude, thanks to the Implicit Function Theorem, that there is a unique differentiable function ψ, defined on some interval centered at $y = -1$, such that $\psi(-1) = 3$ and $F[\psi(y), y] = 0$. Moreover, the derivative $\psi'(y)$ is given there by

$$\psi'(y) = -\frac{F_2[\psi(y), y]}{F_1[\psi(y), y]}. \tag{30}$$

The exression on the right side of equation (30) is precisely what we obtain from implicit differentiation of the equation $F(x, y) = 0$ *with respect to* y while treating x as a function of y.

Now it's easy to see that $\psi'(-1) = 0$, and the curve $x = \psi(y)$ has a tangent parallel to the y-axis at the point $(3, -1)$. But that's a *vertical* tangent.

The reader of this remark can get a better feeling for the difficulties that underlie the naïve "solution" given above by considering the same question for the curve $G(x, y) = 0$ at the point $(0, 0)$, where $G(x, y) = y^2 - x^2$. Is there, or is there not, a vertical tangent to this curve at that point? Where does reasoning as in the "solution" given above lead us?