AP Calculus 2016 AB FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 27, 2017

1 Problem 1

1.1 Part a

We estimate $R'(2)$ as

$$R'(2) \sim \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{2} = -120 \text{ liters/hour}^2. \tag{1}$$

1.2 Part b

To estimate the total amount of water removed from the tank during the time interval $[0, 8]$ with a left Riemann sum having four sub-intervals, we may write

$$R(0) \cdot [1 - 0] + R(1) \cdot [3 - 1] + R(3) \cdot [6 - 3] + R(6) \cdot [8 - 6] = \tag{2}$$

$$1340 \cdot 1 + 1190 \cdot (3 - 1) + 950 \cdot (6 - 3) + 740 \cdot (8 - 6) = 8050. \tag{3}$$

The function R is decreasing, so the left-hand endpoint of each subinterval gives the maximum value of R on that subinterval. Thus, a left-hand Riemann sum gives an overestimate of the integral.

1.3 Part c

The total amount of water in the tank at time t is

$$50000 + \int_0^t [W(\tau) - R(\tau)]\, d\tau = 50000 + 2000 \int_0^t e^{-\tau^2/20}\, d\tau - \int_0^t R(\tau)\, d\tau, \tag{4}$$

or, when $t = 8$,

$$\sim 50000 + 2000 \int_0^8 e^{-\tau^2/20}\, d\tau - 8050. \tag{5}$$

Thus, after carrying out the remaining integration numerically, we find that the amount of water in the tank when $t = 8$ is approximately 49786.19532 liters. To the nearest liter, this is 49786 liters.

1.4 Part d

We consider the function $F(t) = W(t) - R(t)$. The functions W and R are both continuous on the interval $[0, 8]$, so the function F is also continuous on that interval. We have $F(0) = 660$, while $F(8) \sim -618.5$ to the nearest tenth. Thus, $F(0) > 0$ while $F(8) < 0$, and, by the Intermediate Value Propery of continouus functions, there is a point ξ somewhere in the interval $(0, 8)$ for which $F(\xi) = 0$. For this ξ we have $W(\xi) - R(\xi)$, so the answer to the question is "Yes."

2 Problem 2

2.1 Part a

If

$$v(t) = 1 + 2\sin\frac{t^2}{2}, \tag{6}$$

then speed, $\sigma(t)$, is given by

$$\sigma(t) = |v(t)| = \sqrt{[v(t)]^2} = \sqrt{\left[1 + 2\sin\frac{t^2}{2}\right]^2}, \tag{7}$$

so that

$$\sigma('t) = \frac{\left[1 + 2\sin\frac{t^2}{2}\right]\left[2t\cos\frac{t^2}{2}\right]}{\sqrt{\left[1 + 2\sin\frac{t^2}{2}\right]^2}} \tag{8}$$

Now $1 + 2\sin 8\ \sim 2.98 > 0$, so some calculation gives

$$\sigma'(4) = 8\cos 8 \sim -1.164 < 0. \tag{9}$$

Consequently, speed is decreasing when $t = 4$.

2.2 Part b

We seek values of t in $[0,3]$ where $v(t)$ changes sign. This can happen only where $v(t) = 0$ or where

$$0 = 1 + 2\sin\frac{t^2}{2}, \text{ which is} \tag{10}$$

$$\sin\frac{t^2}{2} = -\frac{1}{2}. \tag{11}$$

If $0 \le t \le 3$, then $0 \le t^2 \le 9$, so the only value of t in $[0,3]$ for which (11) can hold is $t = \sqrt{7/\pi/6} \sim 2.07047$. The quantity $t^2/2$ increases through $7\pi/6$ as t increases through $\sqrt{7\pi/6}$, so the value of the sine function at $t^2/2$ decreases through $-1/2$. Thus, $v(t)$ does indeed change sign (from positive to negative) at $t = \sqrt{7\pi/6}$, and this is the only point in the interval $[0,3]$ where there is a sign change for v.

2.3 Part c

By the Fundamental Theorem of Calculus, the position $x(t)$ of the particle at time t is

$$x(t) = x(4) + \int_4^t v(\tau)\,d\tau \tag{12}$$

$$= 2 + \int_4^t \left[1 + 2\sin\frac{\tau^2}{2}\right] d\tau. \tag{13}$$

Setting $t = 0$ and integrating numerically (the integral is not elementary, and we have no other choice), we obtain

$$x(0) = 2 + \int_4^0 \left[1 + 2\sin\frac{\tau^2}{2}\right] d\tau \sim -3.18503. \tag{14}$$

2.4 Part d

The total distance the particle travels during the interval $[0,3]$ is

$$\int_0^3 |v(\tau)|\,d\tau = \int_0^3 \left|1 + 2\sin\frac{\tau^2}{2}\right| d\tau \sim 5.30120, \tag{15}$$

where we have again integrated numerically.

3 Problem 3

For a graph of g, see Figure 1.

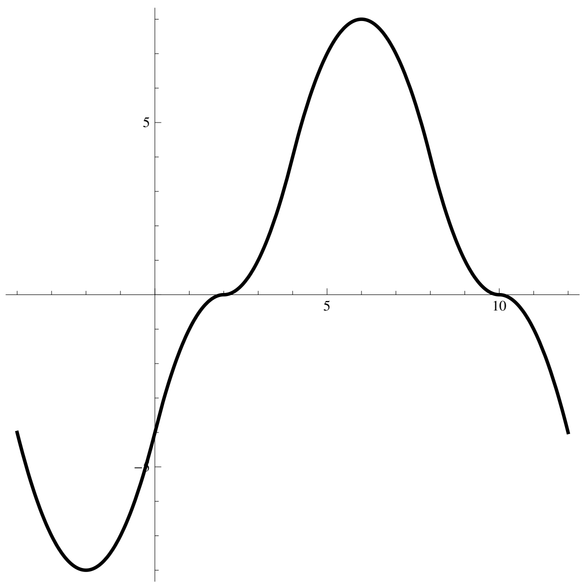

Figure 1: Problem 3, Graph of g

3.1 Part a

If $g(x) = \int_2^x f(t)\,dt$ then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$. While $g'(10) = -$, we see that $g'(x)$ is negative for all values of x in some punctured neighborhood of $x = 10$. Thus, by the First Derivative Test, g has neither a relative minimum nor a relative maximum at $x = 10$.

3.2 Part b

Arguing again from the given graph, which is that of g', we see that g' is increasing on an interval just to the left of $x = 4$ but decreasing on an interval just to the right of $x = 4$. Thus, g has an inflection point where $x = 4$. (In fact, g is concave upward immediately to the left of $x = 4$ and concave downward immediately to the right of $x = 4$.)

3.3 Part c

The absolute minimum value must occur either at an endpoint of the interval or at a point where $g'(x)$ undergoes a sign change from negative to positive as x increases. The only points that qualify are $x = -4$, $x = -2$, and $x = 12$. Summing the areas of the appropriate triangles (with appropriate signs), we see that $g(-4) = -4$, $g(-2) = -9$, and $g(12) = -4$. Thus, g has its absolute minimum at $x = -8$.

Similar reasoning shows that the absolute maximum of $g(x)$ can only be at $x = -4$, $x = 6$, or $x = 12$. But this makes $g(6) = 8$ the absolute maximum. (We evaluated the other two possibilities in the preceding paragraph.)

3.4 Part d

On any interval of the form $[x, 2]$, with $-4 \leq x < 2$, the area between the curve $y = f(x)$ and the x-axis, and lying above the x-axis, exceeds that below the x-axis. Thus guarantees that, for such x, $g(x) < 0$.

On the other hand, on any interval of the form $[2, x]$, with $x > 2$, the area of the region bounded by f and below the x-axis doesn't exceed that of the region above the x-axis unless $x > 10$. This means that $g(x) \geq 0$ for $x \leq x \leq 10$, and $g(x) < 0$ when $10 < x$.

The desired intervals are $[-4, 2]$ and $[10, 12]$.

4 Problem 4

4.1 Part a

See Figure 2.

4.2 Part b

If a solution, f, of the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x-1}$ passes through the point $(2, 3)$, the slope of its tangent line at that point is

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \left.\frac{y^2}{x-1}\right|_{(2,3)} = 9. \tag{16}$$

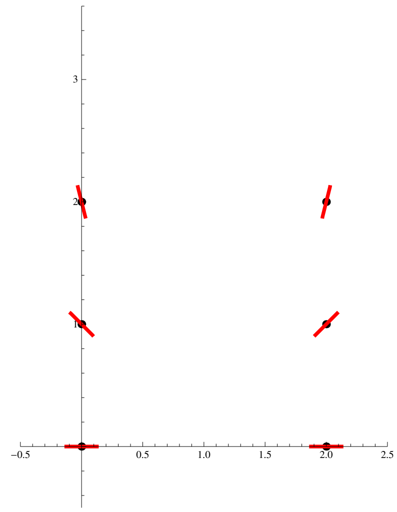

Figure 2: Problem 4a, the slope field

An equation for that tangent line is therefore

$$y = 3 + 9(x - 2). \tag{17}$$

It follows from this that

$$f(2.1) \sim 3 + 9(2.1 - 2) = 3.9. \tag{18}$$

4.3 Part c

Let $y = f(x)$ be the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x-1}$ which passes through the point $(2, 3)$. Then

$$f'(x) = \frac{\big[f(x)\big]^2}{x-1} \tag{19}$$

for all values of x in some open interval I that contains $x = 2$. We may assume that the number one doesn't lie in I, so that $x - 1 > 0$ throughout I. Moreover, because f must be a continuous function whose value at $x = 2$ is 3, we may also assume that $f(x) \neq 0$ anywhere in I. Consequently, throughout I we may write

$$\frac{f'(x)}{\big[f(x)\big]^2} = \frac{1}{x-1}. \tag{20}$$

Moreover, both sides of (20) are integrable on subintervals of I. Consequently, if x is any point of I, we have

$$\begin{aligned}
\int_2^x \frac{f'(\xi)}{\big[f(\xi)\big]^2}\,d\xi &= \int_2^x \frac{d\xi}{\xi-1}; && (21)\\
-\frac{1}{f(\xi)}\bigg|_2^x &= \ln(\xi-1)\bigg|_2^x; && (22)\\
-\frac{1}{f(x)}+\frac{1}{f(2)} &= \ln(x-1)-\ln(2-1); && (23)\\
\frac{1}{3}-\frac{1}{f(x)} &= \ln(x-1). && (24)
\end{aligned}$$

It now follows that

$$f(x) = \frac{3}{1-3\ln(x-1)}. \tag{25}$$

5 Problem 5

5.1 Part a

The average value of the funnel's radius is

$$\begin{aligned}
\frac{1}{10-0}\int_0^{10} \frac{3+h^2}{20}\,dh &= \frac{3}{200}\int_0^{10} dh + \frac{1}{200}\int_0^{10} h^2\,dh && (26)\\
&= \frac{3}{200}\cdot 10 + \frac{1}{200}\cdot\frac{1000}{3} && (27)\\
&= \frac{3}{20}+\frac{5}{3} = \frac{109}{60}. && (28)
\end{aligned}$$

The average value of the radius is $\frac{109}{60}$ inches.

5.2 Part b

The volume, V, of the funnel is

$$V = \pi \int_0^{10} [r(h)]^2 \ dh \tag{29}$$
$$= \frac{\pi}{400} \int_0^{10} (3 + h^2)^2 \, dh \tag{30}$$
$$= \frac{\pi}{400} \int_0^{10} \left(9 + 6h^2 + h^4\right) \ dh \tag{31}$$
$$= \frac{\pi}{400} \left(9h + 2h^3 + \frac{1}{5}h^5\right) \Big|_0^{10} \tag{32}$$
$$= \frac{\pi}{400} (90 + 2000 + 20000) = \frac{2209}{40}\pi \text{ in}^3. \tag{33}$$

5.3 Part c

The radius $r(t)$ and the height $y(t)$ are related by the equation

$$r(t) = \frac{1}{20} \left(3 + [y(t)]^2\right), \tag{34}$$

so that

$$r'(t) = \frac{1}{10} y(t) y'(t), \tag{35}$$

or

$$y'(t) = 10 \frac{r'(t)}{y(t)}. \tag{36}$$

Thus, at the instant when $r'(t_0) = -1/5$ in/sec and $y(t_0) = 3$ in, the height is changing at the rate

$$y'(t_0) = \frac{10}{3} \cdot \left(-\frac{1}{5}\right) = -\frac{2}{3} \text{ in/sec}. \tag{37}$$

6 Problem 6

6.1 Part a

If $k(x) = f[g(x)]$, then $k'(x) = f'[g(x)] \cdot g'(x)$, so

$$k'(3) = f'[g(3)] \cdot g'(3) = f(6) \cdot 2 = 5 \cdot 2 = 10. \tag{38}$$

Also,

$$k(3) = f[g(3)] = f(6) = 4. \tag{39}$$

An equation for the required tangent line is therefore

$$y = k(3) + k'(3)(x - 3), \tag{40}$$

or

$$y = 4 + 10(x - 3). \tag{41}$$

6.2 Part b

If

$$h(x) = \frac{g(x)}{g(x)}, \tag{42}$$

then

$$h'(1) = \frac{g'(1)f(1) - g(1)f'(1)}{[f(1)]^2} \tag{43}$$

$$= \frac{8 \cdot (-6) - 2 \cdot 3}{(-6)^2} = -\frac{3}{2}. \tag{44}$$

6.3 Part c

We let $u = 2x$. Then $du = 2\,dx$ or $dx = \frac{1}{2}du$. Also, $x = 1 \Rightarrow u = 2$, and $x = 3 \Rightarrow u = 6$. Thus

$$\int_1^3 f''(2x)\,dx = \frac{1}{2}\int_2^6 f''(u)\,du \tag{45}$$

$$= \frac{1}{2}f'(u)\Big|_2^6 = \frac{1}{2}\left[f'(6) - f'(2)\right] \tag{46}$$

$$= \frac{1}{2}[5 - (-2)] = \frac{7}{2}. \tag{47}$$