AP Calculus 2017 AB FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 18, 2017

1 Problem 1

1.1 Part a

The approximation with a left-hand sum using the intervals given is

$$50.3 \cdot 2 + 14.4 \cdot 3 + 6.5 \cdot 5 = 176.3. \tag{1}$$

1.2 Part b

We are given that the area of cross-sections decreases as h increases, so the cross-section at the left-hand endpoint of each interval has maximal area for the cross-sections associated with that interval. Thus, the left-hand sum overestimates the volume of the tank. The required approximate volume is 176.3 cubic feet.

1.3 Part c

The volume, in cubic feet, of the tank is $\displaystyle\int_0^{10} \frac{50.3}{e^{0.2h} + h}\,dh$. Numerical integration gives an approximate volume of 101.325 cubic feet for the tank.

1.4 Part d

Let $H(t)$ denote the height of water in the tank at time t. Then the volume, in cubic feet, of water in the tank at time t is

$$V(t) = \int_0^{H(t)} \frac{50.3}{e^{0.2h} + h} \, dh \tag{2}$$

Thus, by the Fundamental Theorem of Calculus and the Chain Rule,

$$V'(t) = \frac{50.3}{e^{0.2H(t)} + H(t)} H'(t), \text{ so that} \tag{3}$$

$$V'(t_0) = 50.3 + \frac{50.3}{e^{0.2H(t_0)} + H(t_0)} H'(t_0), \tag{4}$$

where t_0 is the instant when the depth of water in the tank is five feet. Thus,

$$V'(t_0) = \frac{50.3}{e+5} \cdot 0.26 \sim 1.694 \text{ cubic feet per minute.} \tag{5}$$

2 Problem 2

2.1 Part a

If customers remove bananas from the display at the rate

$$f(t) = 10 + 0.8t \sin\left(\frac{t^3}{100}\right) \tag{6}$$

when $0 < t \leq 12$, then during the interval $0 < t \leq 2$, they have removed $\int_0^2 f(t)\,dt$ pounds of bananas. Integrating numerically, we find that

$$\int_0^2 f(t)\,dt = \int_0^2 \left[10 + 0.8t \sin\left(\frac{t^3}{100}\right)\right] dt \sim 20.051, \tag{7}$$

so that customers have removed about 20.051 pounds of bananas during the first two hours that the store is open.

2.2 Part b

We have

$$f'(t) = 0.024t^3 \cos\left(\frac{t^3}{100}\right) + 0.8 \sin\left(\frac{t^3}{100}\right), \text{ so} \tag{8}$$

$$f'(7) \sim -8.120. \tag{9}$$

Thus, when the store has been open for seven hours, the rate at which customers are removing bananas is decreasing at the rate of 8.120 pounds per hour per hour.

2.3 Part c

When $3 < t \leq 12$, the rate at which the weight of bananas in the display is changing is given by

$$R(t) = -\left[10 + 0.8t \sin\left(\frac{t^3}{100}\right)\right] + 3 + 2.4 \ln\left(t^2 + 2t\right) \tag{10}$$

$$= 2.4 \ln\left(t^2 + 2t\right) - 0.8t \sin\left(\frac{t^3}{100}\right) - 7. \tag{11}$$

Thus,

$$R(5) = -2.263. \tag{12}$$

This rate is negative, so the weight of bananas in the display is decreasing at that time.

2.4 Part d

When $t > 3$, then, the weight $W(t)$ of the bananas in the display is given by

$$W(t) = 50 - \int_0^t \left[10 + 0.8\tau \sin\left(\frac{\tau^3}{100}\right)\right] d\tau + \int_3^t \left[3 + 2.4 \ln\left(\tau^2 + 2\tau\right)\right] d\tau \tag{13}$$

A numerical integration then gives $W(8) \sim 23.347$, so there are 23.347 pounds of bananas in the display when $t = 8$.

3 Problem 3

3.1 Part a

By the Fundamental Theorem of Calculus, $\int_{-6}^{-2} f'(x)\,dx = f(-2) - f(-6) = 7 - f(-6)$. But the value of this integral is the area of a triangle whose base is four and whose altitude is two, so $7 - f(-6) = 4$, and $f(-6) = 3$. Similarly, $\int_{-2}^{5} f'(x)\,dx = f(5) - 7$, while the value of this integral is the area of a triangle of base three, altitude two, less the area of a half disk of radius two. Hence, $f(5) = 7 + 3 - 2\pi = 10 - 2\pi$.

3.2 Part b

The function f is increasing on the closures of those intervals where f' is positive, or on $[-6, -2]$ and on $[2, 5]$.

3.3 Part c

The absolute minimum for f on $[-6, 5]$ must occur either at an endpoint or at a critical point where the derivative changes sign from negative to positive. Thus, the only possibilities are $x = -6$, $x = 2$, and $x = 5$. We already (see Part a, above) have $f(-6) = 3$ and $f(5) = 10 - 2\pi$, which latter is about 3.717, so we need only calculate $f(2)$. But $f(2)$ is less than $f(5)$ by the area of a triangle whose base is three and whose altitude it two, so $f(2) = 7 - 2\pi \sim 0.717$. Now $7 - 2\pi < 3 < 10 - 2\pi$, so the absolute minimum we seek is $f(2) = 7 - 2\pi$.

3.4 Part d

In the vicinity of $x = -5$, the graph of f' is a line whose slope is $-1/2$, so $f''(-5) = -1/2$. Immediately to the left of $x = 3$, the graph of f' is given by a straight line of slope 2, so the left-hand derivative, $f''_{-}(3)$ of f' at $x = 3$ must be 2. Immediately to the right of $x = 3$, the graph of f' is given by a line of slope -1, so the right-hand derivative, $f''_{+}(3)$, of f' at $x = 3$ must be given by $f''_{+}(3) = -1$. The one-sided derivatives of f' at $x = 3$ are different, so $f''(3)$ doesn't exist.

4 Problem 4

4.1 Part a

We have $4H'(t) = 27 - H(t)$; $H(0) = 91$. Thus,

$$H'(0) = \frac{27 - 91}{4} = -16, \tag{14}$$

and an equation for the tangent line at $(0, H(0))$ is $H = 91 - 16t$. Setting $t = 3$ in this equation for the tangent line, we obtain the approximation $H = 43$ for the value of $H(3)$.

4.2 Part b

Differentiating the original equation, we obtain $H''(t) = -H'(t)/4$. Substituting for $H'(t)$ then gives

$$H''(t) = -\frac{1}{4} \cdot \frac{27 - H}{4} = \frac{H - 27}{16}. \tag{15}$$

Thus, $H''(0) = (91 - 27)/16 = 4 > 0$. This means that the solution curve lies above its tangent line near $t = 0$, so we expect our estimate to be an underestimate.

4.3 Part c

If $G'(t) = -[G(t) - 27]^{2/3}$ with $G(0) = 91$, then

$$\frac{G'(t)}{[G(t) - 27]^{3/2}} = -1, \tag{16}$$

and $(G(t) - 27) > 0$ when t is near $t = 0$ because G—being a solution to a differential equation—must be a continuous function. For such values of t, we may therefore write

$$\int_0^t \frac{G'(\tau)}{[G(\tau) - 27]^{2/3}}\, d\tau = -\int_0^t d\tau, \tag{17}$$

so that

$$3[G(t) - 27]^{1/3} - 3[G(0) - 27]^{1/3} = -t, \tag{18}$$

or

$$3[G(t) - 27]^{1/3} = 12 - t. \tag{19}$$

From this it follows that

$$G(t) = \frac{1}{27}(2457 - 432t + 36t^2 - t^3) \tag{20}$$

Thus, $G(3) = 54$, and the internal temperature of the potato at $t = 3$ is, according to this model, 54 degrees Celsius.

5 Problem 5

5.1 Part a

The position of particle P is given by $x_P(t) = \ln(t^2 - 2t + 10)$ when $0 \leq t \leq 8$, so its velocity is given by

$$x'_P(t) = \frac{2t - 2}{t^2 - 2t + 10} = \frac{2(t-1)}{(t-1)^2 + 9}. \tag{21}$$

The denominator of this fraction is never negative, being 9 larger than a square. Therefore, the sign of the derivative (which is velocity) is determined by the sign of its numerator. It follows that $x'_P(t)$ is positive when t lies in $(1, 8]$ and negative when t lies in $[0, 1)$. We conclude that particle P is moving leftward when $0 \leq t < 1$. (For future reference, we note that particle P is moving rightward when $1 < t \leq 8$.)

5.2 Part b

The velocity $v(t)$ of particle Q is given by $v(t) = t^2 - 8t + 15$ on $[0, 8]$. But

$$\begin{aligned} v(t) &= t^2 - 8t + 15 & (22) \\ &= (t-5)(t-3). & (23) \end{aligned}$$

It follows from analysis of the sign of this last quantity that particle Q is moving to the right when t lies in either of the intervals $[0, 3)$ or $(5, 8]$, and that particle Q is moving to the left when t lies in the interval $(3, 5)$.

Comparing this with what we learned (in Part a, above) about particle P's motion, we see that the two particles are both moving rightward when t satisfies $1 < t < 3$ and when t satisfies $5 < t \leq 8$. At no time are they both moving leftward.

5.3 Part c

The acceleration of particle Q is

$$v'(t) = 2t - 8, \tag{24}$$

so $v'(2) = -4$.

Speed, $\sigma(t) = |v(t)|$, can be written $\sigma(t) = \sqrt{[v(t)]^2}$, so when $t \neq 0$ we may take the derivative with respect to t to get

$$\sigma'(t) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}v(t)v'(t)}{\sqrt{[v(t)]^2}} = \frac{v(t)}{|v(t)|}v'(t). \tag{25}$$

Using what we have seen in earlier parts of the problem,

$$\begin{align} \sigma'(2) &= \frac{v(2)}{|v(2)|}v'(2) \tag{26} \\ &= \frac{(2-5)(2-3)}{|(2-5)(2-3)|} \cdot (-4) \tag{27} \\ &= \frac{3}{|3|} \cdot (-4) < 0, \tag{28} \end{align}$$

We see, then, that speed is decreasing when $t = 2$ because $\sigma'(2) < 0$.

5.4 Part d

We have seen (in Part b, above) that particle Q first changes direction when $t = 3$. The position, $x(t)$, of particle Q at time t satisfies, by the Fundamental Theorem of Calculus,

$$\begin{align} x(t) - x(0) &= \int_0^t x'(\tau)\, d\tau = \int_0^t v(\tau)\, d\tau, \text{ so} \tag{29} \\ x(3) &= x(0) + \int_0^3 v(\tau)\, d\tau \tag{30} \\ &= 5 + \int_0^3 \left(\tau^2 - 8\tau + 15\right)\, d\tau \tag{31} \\ &= 5 + \left[\frac{\tau^3}{3} - 4\tau^2 + 15\tau\right]\Bigg|_0^3 = 23. \tag{32} \end{align}$$

Thus, the position $x(3)$ of particle Q when $t = 3$ is $x(3) = 23$.

6 Problem 6

6.1 Part a

Because

$$f(x) = \cos(2x) + e^{\sin x}, \tag{33}$$

we must have

$$f'(x) = -2\sin(2x) + e^{\sin x}\cos x. \tag{34}$$

Consequently,

$$f(\pi) = \cos(2\pi) + e^{\sin \pi} = 2 \tag{35}$$

and

$$f'(\pi) = -2\sin(2\pi) + e^{\sin \pi}\cos \pi = -1. \tag{36}$$

The slope of the desired tangent line is therefore -1 and an equation is

$$y = 2 - (x - \pi). \tag{37}$$

6.2 Part b

If $k(x) = h[f(x)]$, then $k'(\pi) = h'[f(\pi)]\, f'(\pi) = h'(2) \cdot (-1)$, where the latter equality arises from equations (35) and (36). From the graph of h, we see that

$$h'(2) = -1/3, \tag{38}$$

the relevant portion of that graph being a line segment connecting $(0, 0)$ and $(3, -1)$ and having slope $(-1 - 0)/[3 - 0] = -1/3$. Hence $k'(\pi) = 1/3$.

6.3 Part c

If $m(x) = g(-2x) \cdot h(x)$, then

$$m'(x) = -2 \cdot g'(-2x) \cdot h(x) + g(-2x) \cdot h'(x), \tag{39}$$

so that

$$m'(2) = -2 \cdot g'(-4) \cdot h(2) + g(-4) \cdot h'(2). \tag{40}$$

From the table, we see that $g(-4) = 5$ and $g'(-4) = -1$. From the graph, we read $h(2) = -2/3$, and from equation (38) we have $h'(2) = -1/3$. Thus,

$$m'(2) = (-2) \cdot (-1) \cdot (-\frac{2}{3}) + (5) \cdot (-\frac{1}{3}) = -\frac{9}{3}. \tag{41}$$

6.4 Part d

We are given that g is "a differentiable function," but we aren't told where, so we will assume that g is differentiable wherever we want it to be—on some open interval containing the interval $[-5, -3]$ in particular. This guarantees that g is continuous on $[-5, -3]$, and differentiable on $(-5, -3)$. We may therefore apply the Mean Value Theorem to g on that interval to conclude that there is a number, c, in $(-5, -3)$, and therefore in $[-5, -3]$, such that

$$g'(c) = \frac{g(-5) - g(-3)}{(-5) - (-3)} = \frac{(10) - (2)}{(-5) - (-3)} = -4. \tag{42}$$

Remark: If we assume that this last question can be answered with the tools of elementary calculus, then the answer must be "Yes." That's because failure of the Mean Value Theorem's guarantee doesn't show that there can be no value c that satisfies the conclusion of that theorem.