AP Calculus 2019 AB FRQ Solutions

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1 Problem 1

1.1 Part a

Because fish enter the lake at the rate $E(t) = 20 + 15\sin\frac{\pi t}{6}$, with t in hours after midnight, the number, F, of fish that enter the lake over the 5-hour period from $t = 0$ to $t = 5$, is given by

$$F = \int_0^5 E(t)\,dt = \int_0^5 \left[20 + 15\sin\frac{\pi\tau}{6}\right]\,d\tau \tag{1}$$

$$= \left[20\tau - 90\cos\frac{\pi\tau}{6}\right]\Bigg|_0^5 = 100 + \frac{45(2+\sqrt{3})}{\pi} \sim 153, \tag{2}$$

to the nearest whole number.

1.2 Part b

In like fashion, the average number $\overline{F}$ of fish that leave the lake per hour over the same 5-hour period is given by

$$\overline{F} = \frac{1}{5}\int_0^5 \left(4 + 2^{0.1\tau^2}\right)\,d\tau \sim 6.059. \tag{3}$$

The problem statement givess no instruction to give this answer to the nearest whole number, so we have given it through the first three decimal digits.

1.3 Part c

Let F_0 be the number of fish in the lake at time $t = 0$. The number $f(t)$ of fish in the lake at time t, for $0 \leq t \leq 8$, is then given by

$$f(t) = F_0 + \int_0^t \left[16 + 15 \sin \frac{\pi\tau}{6} - 2^{0.1\tau^2}\right] d\tau \tag{4}$$

By the Fundamental Theorem of Calculus,

$$f'(t) = 16 + 15 \sin \frac{\pi t}{6} - 2^{0.1t^2}. \tag{5}$$

Setting $f'(t) = 0$ and solving numerically, we find just one critical point in $[0, 8]$ at $t_0 \sim 6.204$. The maximum value of f on $[0, 8]$ occurs at an endpoint of the interval or at a critical point, so it must therefore be found at one of three points: at $t = 0$, $t = t_0$, or $t = 8$. Evaluating, gives

$$f(0) = F_0; \tag{6}$$

$$f(t_0) \sim F_0 + 135.015; \tag{7}$$

$$f(8) = F_0 + 80.920. \tag{8}$$

We conclude that the number of fish in the lake is greatest when $t = t_0 \sim 6.204$.

1.4 Part d

As we have seen in Part c of this problem, the rate of change of the number of fish in the lake during the relevant interval is

$$f'(t) = 16 + 15 \sin \frac{\pi t}{6} - 2^{0.1t^2}. \tag{9}$$

Thus,

$$f''(t) = \frac{5\pi}{2} \cos \frac{\pi t}{6} - \frac{2^{0.1t^2}}{5} t \ln 2. \tag{10}$$

So $f''(5) \sim -10.723$, and the rate of change of the number of fish in the lake is decreasing at about the rate of 10.723 fish per hour per hour.

2 Problem 2

2.1 Part a

The function v_P is given differentiable, presumably on some interval containing the interval $(0, 4)$, although this isn't stated. Consequently v_P is differentiable on the interval $(0.3, 2.8)$ and continuous on the interval $[0.3, 2.8]$. It is also given that $v_P(0.3) = v_P(2.8) = 55$. The Mean Value Theorem applies to guarantee the existence of a point ξ in $(0.3, 2.8)$ such that v'_P, which is the acceleration of the particle P satisfies

$$2.5\, v'_P(\xi) = v'_P(\xi)(2.8 - 0.3) = [v_P(2.8) - v_P(0.3)] = 55 - 55 = 0. \tag{11}$$

It is immediate from this that $v'_P(\xi) = 0$.

Note that it is also possible to solve this problem by appealing to Rolle's Theorem instead of to the Mean Value Theorem.

2.2 Part b

The trapezoidal rule with the subintervals $[0, 0.3]$, $[0.3, 1.7]$, and $[1.7, 2.8]$ gives

$$\int_0^{2.8} v_P(t)\, dt \sim \frac{1}{2}\big[(55+0)(0.3-0) + (-29+55)(1.7-0.3) + (55-29)(2.8-1.7)\big] \tag{12}$$
$$\sim 40.75. \tag{13}$$

2.3 Part c

We are to find the subinterval of $[0, 4]$ for which $v_Q(t) \geq 60$, where

$$v_Q(t) = 45\sqrt{t}\cos(0.063t^2), \tag{14}$$

so we must solve the inequality

$$60 \leq 45\sqrt{t}\cos(0.063t^2), \text{ or, equivalently,} \tag{15}$$
$$\frac{4}{3} \leq \sqrt{t}\cos(0.063t^2). \tag{16}$$

Examination of a graph indicates that there is two instances of equality in $[0, 4]$, and that the solution consists of the interval whose endpoints are those two values. Solving numerically, we find that the desired interval is, approximately, $[1.866, 3.519]$. If S_Q is the distance this particle travels during this interval, then

$$S_Q \sim \int_{1.866}^{3.510} |v_Q(t)|\, dt \tag{17}$$
$$\sim \int_{1.866}^{3.510} 45\sqrt{t}\cos(0.063t^2)\, dt \sim 135.938 \text{ meters.} \tag{18}$$

2.4 Part d

The integral of part (c) gives the displacement of the particle P at time $t = 2.8$, while the displacement of particle Q at that time is given by

$$-90 + \int_0^{2.8} v_Q(t)\, dt \sim 73.067 \text{ meters.} \tag{19}$$

The distance between the two particles at time $t = 2.8$ is the magnitude of the difference between these two displacements, or about 95.188 meters.

3 Problem 3

From what is given about the function f, we easily write

$$f(x) = \begin{cases} -x - 1, & -2 \le x \le 0; \\ \\ 2x - 1, & 0 \le x \le 2; \\ \\ 3 - \sqrt{9 - (x-5)^2}, & 2 \le x \le 5. \end{cases} \tag{20}$$

It is also given that the domain of f is the interval $[-6, 5]$ and that f is continuous on that interval.

3.1 Part a

From the properties of the definite integral, we know that

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx. \tag{21}$$

But it is given that

$$\int_{-6}^{5} f(x)\,dx = 7. \tag{22}$$

We have

$$\int_{-2}^{5} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \tag{23}$$

$$= 0 + 2 + \left(9 - \frac{9}{4}\pi\right) = 11 - \frac{9}{4}\pi. \tag{24}$$

We can carry out the integrations of (23) by using (20), or we can carry them out by using the geometry of the graph to find the areas of several triangles, a square, and a quarter-circle.

Finally,

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dxdx \tag{25}$$

$$= 7 - \left(11 - \frac{9}{4}\pi\right) = \frac{9}{4}\pi - 4. \tag{26}$$

3.2 Part b

We have

$$\int_3^5 [2f'(x)+4]\ dx = 2\int_3^5 f'(x)\,dx + 4\int_3^5 dx \tag{27}$$
$$= 2\,[f(5)-f(3)] + 4(5-3) \tag{28}$$
$$= 2\left[0-\left(3-\sqrt{5}\right)\right] + 8 = 2+2\sqrt{5}. \tag{29}$$

3.3 Part c

The function $g(x) = \int_{-2}^{x} f(t)\,dt$ gives the signed area between the curve $y = f(t)$ and the t-axis on the interval $[-2,x]$. It is visually evident that the area between the curve and the horizontal axis on the interval $[-2,-1]$ is a small positive number (in fact, it is $1/2$), the area between the curve and the horizontal axis on the interval $[-1,1/2]$ is a negative number of small magnitude (in fact, it is $-3/4$), and the area between the curve and the horizontal axis on the interval $[-1/2,5]$ is a positive number substantially larger than either of the other two magnitudes. So $g(x)$ increases on the interval $[-2,-1]$ from 0 to $g(-2)=0$ to $g(-1)=1/2$. On the interval $[-1,1/2]$, the value $g(x)$ decreases from $g(-1)=1/2$ to $g(1/2)=-1/4$, and on the interval $[1/2,5]$ the value $g(x)$ increases from $1/4$ to a relatively large positive number (in fact, to $g(5)=11-9\pi/4$, as we saw in the course of our solution to Part a of this problem.) It follows from these considerations that the maximum value of g on $[-2,5]$ is $g(5)=11-9\pi/4$

We are given that f is continuous, so $\lim_{x\to 1} f(x) = f(1) = 1$. We are given that the graph of f is a straight line of slope 2 on the interval $[0,2]$ so $f'(x)\equiv 2$ on $(0,2)$. Hence, $\lim_{x\to 1} f'(x) = 2$. Also, $\lim_{x\to 1} 10^x = 10$ and $\lim_{x\to 1} \arctan x = \pi/4$, because both of these functions are continuous at $x=1$. Where continuous functions are involved, the limit of a difference is the difference of a limit, and the limit of a quotient is the quotient of the limits (provided the limit in the denominator is not zero). Thus,

$$\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x)-\arctan x} = \frac{10-3\cdot 2}{1-\pi/4} = \frac{16}{4-\pi}. \tag{30}$$

4 Problem 4

4.1 Part a

Now $h'(t) = -\sqrt{h}/10$, and $V = \pi r^2 h = \pi h$ (because r is constant), and this means that

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} \tag{31}$$
$$= \pi\cdot\left(-\frac{\sqrt{h}}{10}\right) \text{ cubic feet per second.} \tag{32}$$

4.2 Part b

We have

$$\frac{d}{dt}\left(\frac{dh}{dt}\right) = -\frac{d}{dt}\left(\frac{\sqrt{h}}{10}\right) = -\left(\frac{1}{20\sqrt{h}}\right)\frac{dh}{dt} \tag{33}$$

$$= -\left(\frac{1}{20\cancel{\sqrt{h}}}\right)\cdot\left(-\frac{\cancel{\sqrt{h}}}{10}\right) = \frac{1}{200}. \tag{34}$$

We conclude that $h'(t)$, which is the rate at which the rate of change of h changes, has a positive derivative at all times, and so is always increasing.

4.3 Part c

Let $h(t)$ designate the solution we seek. We are given that $h'(t) = -\sqrt{h(t)}/10$ along with the initial conditon $h(0) = 5$. Hence,

$$\frac{h'(t)}{\sqrt{h(t)}}\,dt = -\frac{1}{10},\ \text{so} \tag{35}$$

$$\int_0^t \frac{h'(\tau)}{\sqrt{h(\tau)}}\,d\tau = -\int_0^t \frac{d\tau}{10}. \tag{36}$$

We know that $h(0) = 5 > 0$, so the solution is positive at $t = 5$. Being differentiable (as a solution of a differential equation) is must be continuous near $t = 5$, and so must be positive in some neighborhood of 5.. Integrating (36) to some value of t where $h(t)$ remains positive, we find that

$$2\sqrt{h(t)} - 2\sqrt{h(0)} = -\frac{t}{10}, \tag{37}$$

which, using the initial condition, becomes

$$\sqrt{h(t)} = \sqrt{5} - \frac{t}{20} \tag{38}$$

We can rewrite this as

$$h(t) = 5 - \frac{\sqrt{5}}{10}t + \frac{t^2}{400}. \tag{39}$$

5 Problem 5

R denotes the region enclosed by the graphs of $g(x) = -2 + 3\cos(\pi x/2)$ and $h(x) = 6 - 2(x-1)^2 = 4 + 4x - 2x^2$, the y-axis, and the vertical line $x = 2$.

5.1 Part a

The area of the region R is

$$\int_0^2 [h(x) - g(x)]\,dx = \int_0^2 \left[6 + 4x - 2x^2 - 3\cos\frac{\pi x}{2}\right]\,dx \tag{40}$$

$$= \left[6x + 2x^2 - \frac{2}{3}x^3 - \frac{6}{\pi}\sin\frac{\pi x}{2}\right]\Bigg|_0^2 \tag{41}$$

$$= \frac{44}{3}. \tag{42}$$

5.2 Part b

The area of the cross-section at $x = t$ is given as

$$A(t) = \frac{1}{t+3} \tag{43}$$

for x extending from 0 to 2, so the volume in question is

$$V = \int_0^2 A(t)\,dt = \int_0^2 \frac{dt}{t+3} = \ln|t+3|\Bigg|_0^2 = \ln\frac{5}{3}. \tag{44}$$

5.3 Part c

The volume, V generated when R is rotated about the line $y = 6$ is

$$V = \pi\int_0^2 \left([6 - g(x)]^2 - [6 - h(x)]^2\right)dx \tag{45}$$

$$= \pi\int_0^2 \left[9\cos^2\frac{\pi x}{2} - 48\cos\frac{\pi x}{2} - 4x^4 + 16x^3 - 24x^2 + 16x + 60\right]\,dx \tag{46}$$

Evaluation of this integral is not required. However it is an elementary integral, and for those who *must* know,

$$V = \frac{677\pi}{5}. \tag{47}$$

6 Problem 6

6.1 Part a

If the line $y = 4 + \frac{2}{3}(x-2)$, which has slope $\frac{2}{3}$, is tangent to the curve $y = h(x)$ at $x = 2$, then $h'(2) = \frac{2}{3}$ because $h'(2)$ is the slope of the line tangent to $y = h(x)$ at $x = 2$. For future reference, we note that this tangent line must pass through the point with coordinates $\big(2, h(2)\big)$. The point on the line where $x = 2$ being the point with coordinates $(2, 4)$, we would, if it had not been given, conclude that $h(2) = 4$.

6.2 Part b

If $a(x) = 3x^3 h(x)$, then, by the Chain Rule,

$$a'(x) = 9x^2 h(x) + 3x^3 h'(x), \text{ whence} \tag{48}$$

$$a'(2) = 9 \cdot 2^2 \cdot 4 + \cancel{3} \cdot 2^3 \cdot \frac{2}{\cancel{3}} \tag{49}$$

$$= 144 + 16 = 160. \tag{50}$$

(We have used the values of $h(2)$ and $h'(2)$ from Part a.)

6.3 Part c

If it is known that the limit, as $x \to 2$ of $\dfrac{x^2 - 4}{1 - [f(x)]^3}$ can be evaluated using l'Hôpital's Rule, then—because $x^2 - 4 \to 0$ as $x \to 2$—it must be that $1 - [f(x)]^3 \to 0$ as $x \to 2$; this is because l'Hôpital's rule can't be used unless the limits in numerator and denominator both be zero or both be infinite. But f is given twice differentiable, so f and f' must both be continuous. Consequently, $1 - [f(2)]^3 = 0$, and it follows that $f(2) = 1$. But, because $2x \to 4$, which is neither 0 nor infinite, (This must be noted in order to ensure that only a single application of l'Hôpital's rule is needed, and this fact has not been given.) use of l'Hôpital's rule also requires—using the continuity of f and of f', that

$$\lim_{x \to 2} \frac{2x}{3[f(x)]^2 f'(x)} = \frac{4}{3[f(2)]^2 f'(2)}. \tag{51}$$

This quotient is $\dfrac{4}{3 \cdot 1 \cdot f'(2)}$. But, by the continuity of h, which is given, and the fact that $h(x)$ agrees with the fraction $\dfrac{x^2 - 4}{1 - [f(x)]^3}$ when $x \neq 2$, we now see that

$$4 = h(2) = \frac{4}{3f'(2)}, \tag{52}$$

from which it is immediate that

$$f'(2) = \frac{16}{3}. \tag{53}$$

6.4 Part d

The functions g and h are both continuous (both being twice differentiable), and both take on the value 4 at $x = 2$. Both, therefore, have 4 as their limiting values when $x \to 2$. But from $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$ it follows that

1.

$$4 = g(2) \leq k(2) \leq h(2) = 4, \text{or} \tag{54}$$
$$k(2) = 4, \tag{55}$$

and that

2.

$$4 = \lim_{x \to 2} g(x) \leq \lim_{x \to 2} k(x) \leq \lim_{x \to 2} h(x) = 4, \text{ or} \tag{56}$$
$$\lim_{x \to 2} k(x) = 4. \tag{57}$$

Hence $k(2) = 4 = \lim_{x \to 2} k(x)$, from which we conclude that k is continuous at $x = 2$.

(Note: In place of the compound inequality (56), one can apply the "Squeeze Theorem," which is also known as the "Sandwich Theorem," the "Flyswatter Principle," and, probably, many other names.)