AP Calculus 2021 AB FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 11, 2021

1. **Solution:** We begin by writing $r_0 = 0$, $r_1 = 1$, $r_2 = 2$, $r_3 = 2.5$, and $r_4 = 4$.

   (a) To approximate $f'(2.25)$ from the given data, we write

   $$f'(2.25) \sim \frac{f(2.5) - f(2.0)}{2.5 - 2.0} = \frac{10 - 6}{2.5 - 2.0} = \frac{4}{0.5} = 8. \tag{1}$$

   Thus, $f(2.25) \sim 8$ mg/cm^3. This means that as we move directly outward from the center of the Petri dish, the density of bacteria is increasing at about the per-centimeter-rate of 8 milligrams per square centimeter at a point 2.25 centimeters from the center.

   (b) The required right Riemann sum tp approximate $2\pi \displaystyle\int_0^4 r f(r)\,dr$ is

   $$2\pi \sum_{k=1}^{4} r_k f(r_k)(r_k - r_{k-1}) = 2\pi\,(1 \cdot 2 \cdot 1 + 2 \cdot 6 \cdot 1 + 2.5 \cdot 10 \cdot 0.5 + 4 \cdot 18 \cdot 1.5) \text{ mg.} \tag{2}$$

   This is 269π mg.

   (c) It is given that f is an increasing function. It follows that for any integer $k = 1, 2, 3, 4$ and $r_{k-1} \le r < r_k$, then $r f(r) < r f(r_k) < r_k f(r_k)$. Hence, for all such k, we must have

   $$\int_{r_{k-1}}^{r_k} r f(r)\,dr < \int_{r_{k-1}}^{r_k} r_k f(r_k)\,dr < r_k f(r_k)(r_k - r_{k-1}). \tag{3}$$

   We now see that

   $$\int_0^4 r f(r)\,dr = \sum_{k=1}^{4} \int_{r_{k-1}}^{r_k} r f(r)\,dr < \sum_{k=1}^{4} r_k f(r_k)(r_k - r_{k-1}). \tag{4}$$

   The right Riemann sum is therefore an overestimate for the corresponding integral.

(d) The average value of g on $[1, 4]$ is

$$\frac{2\pi}{4} \int_1^4 g(r)\, dr = \pi \int_1^4 \left[1 - 8\cos^3(1.57\sqrt{r})\right] dr. \tag{5}$$

Carrying out a numerical integration, we find that the required average value is about 44.186 milligrams per square centimeter.

2. **Solution:**

(a) By the Fundamental Theorem of Calculus, the positions $x_P(t)$ and $x_Q(t)$ of, respectively, particles P and Q at time t are given by

$$x_P(t) = x_P(0) + \int_0^t v_P(\tau)\, d\tau \tag{6}$$

$$= 5 + \int_0^t \sin \tau^{1.5}\, d\tau \tag{7}$$

and

$$x_Q(t) = x_Q(0) + \int_0^t v_Q(\tau)\, d\tau \tag{8}$$

$$= 10 + \int_0^t \left[(\tau - 1.8) + 1.25^\tau\right] d\tau \tag{9}$$

Integrating numerically with $t = 1$ gives $x_P(1) \sim 5.371$ and $x_Q(1) \sim 9.820$.

(b) The distance $D(t)$ between P and Q at time t is

$$D(t) = \left|x_Q(t) - x_P(t)\right|, \tag{10}$$

which is always non-negative. Moreover,

$$\left[D(t)\right]^2 = \left[x_Q(t) - x_P(t)\right]^2. \tag{11}$$

Therefore,

$$2D(t)D'(t) = 2\left[x_Q(t) - x_P(t)\right]\left[x'_Q(t) - x'_P(t)\right], \tag{12}$$

Thus,

$$D(1)D'(1) = \left[x_Q(1) - x_P(1)\right]\left[v_Q(1) - v_P(1)\right]. \tag{13}$$

Now, from our calculations in part (a),

$$x_Q(1) - x_P(1) \sim 9.82036 - 5.37066 \sim 4.450, \tag{14}$$

while

$$v_Q(1) - v_P(1) \sim [(1 - 1.8) + 1.25^1] - \sin 1 \sim 0.45000 - 0.84147 \sim -0.391 \tag{15}$$

Now $D(1) > 0$, and it follows from (12) and the definitions of x_P, x_Q, and D that D' is a continuous function whereever D does not vanish. So from (13) and what we have just seen, we conclude that D is a decreasing function in the vicinity of $t = 1$ because its derivative is negative there and functions whose derivatives are negative on an interval must be decreasing on that interval. When $t = 1$, the two particles are moving toward from each other.

(c) The acceleration of particle Q is $v'_Q(t) = 1 + 1.25^t \ln 1.25$. Thus, the acceleration of particle Q is $v'_Q(1) \sim 1.279$. The speed, $s_Q(t)$ of particle Q at time t is $|v_Q(t)|$, which is a non-negative quantity that satisfies

$$\left[s_Q(t)\right]^2 = [v_Q(t)]^2, \tag{16}$$

whence

$$2s_Q(t)s'_Q(t) = 2v_Q(t)v'_Q(t). \tag{17}$$

It follows that, wherever speed does not vanish, the sign of $s'_Q(t)$ is the same as that of $v_Q(t)v'_Q(t)$. Now $v_Q(1)v'_Q(1) \sim (0.450)(1.279) > 0$, so speed is increasing near $t = 1$.

(d) The total distance traveled by particle P during the time interval $0 \le t \le \pi$ is

$$\int_0^\pi |v_P(\tau)|\, d\tau = \int_0^\pi \sin \tau^{1.5}\, d\tau \tag{18}$$

Integrating numerically, we find that the distance is approximately 0.515.

3. **Solution:**

(a) The area in the first quadrant bounded by the x-axis and the curve $y = 6x\sqrt{4 - x^2}$ is

$$3\int_0^2 \sqrt{4 - x^2} \cdot 2x\, dx = -2(4 - x^2)^{3/2}\Big|_0^2 = -0 + 16 = 16 \text{ in}^2 \tag{19}$$

(b) If y is as above, $y' = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}}$ and $y' = 0$ for $0 \le x \le 2$, then $x = \sqrt{2}$. We are given $c > 0$, so because $y = 0$ when $x = 0$ or $x = 2$ and $y > 0$ when $0 < x < 2$, we see that y assumes its absolute minimum on $[0, 2]$ at the endpoints. Applying the Extreme Value Theorem y, which depends continuously on x throughout the closed, bounded interval $[0, 2]$, must have an absolute maximum somewhere interior to that interval. By Fermat's Theorem that maximum must occur at a value of x where $y' = 0$. There being only one such value, it must yield the maximum. Because y gives, for each x, the radius of the corresponding cross-sectional slice, we conclude that

$$1.2 = c\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2c. \tag{20}$$

It follows that $c = 0.6$

(c) The volume of the spinning toy generated by the curve $y = cx\sqrt{4-x^2}$ is

$$c^2\pi \int_0^2 x^2(4-x^2)\,dx = c^2\pi \int_0^2 (4x^2 - x^4)\,dx \tag{21}$$

$$= c^2\pi \left(4\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^2 = c^2\pi \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64}{15}c^2\pi. \tag{22}$$

If this is to be 2π, we must have $c = \dfrac{\sqrt{30}}{8}$.

4. **Solution:**

(a) If $G(x) = \displaystyle\int_0^x f(t)\,dt$, then, according to the Fundamental Theorem of Calculus, $G'(x) = f(x)$. Now G is concave upward on those open intervals where $G'(x) = f(x)$ is increasing. Because we see from its graph that f is increasing on $[-4, -2]$ and on $[2, 6]$, we conclude that G is concave upward on $(-4, -2)$ and on $(2, 6)$.

(b) If $P(x) = G(x) \cdot f(x)$, then

$$P'(x) = G'(x) \cdot f(x) + G(x) \cdot f'(x), \tag{23}$$

so

$$P'(3) = G'(3) \cdot f(3) + G(3) \cdot f'(3) \tag{24}$$

$$= f(3) \cdot f(3) + G(3) \cdot f'(3). \tag{25}$$

Now

$$f(3) = -3, \tag{26}$$

$$f'(3) = \frac{f(6) - f(2)}{6-2} = \frac{0-(-4)}{6-2} = 1, \tag{27}$$

and

$$G(3) = \frac{f(0)+f(2)}{2} \cdot 2 + \frac{f(2)+f(3)}{2} \cdot 1 \tag{28}$$

$$= \frac{4+(-4)}{\cancel{2}} \cdot \cancel{2} + \frac{-4+(-3)}{2} \cdot 1 = -\frac{7}{2} \tag{29}$$

so

$$P'(3) = (-3)^2 + 1 \cdot \left(-\frac{7}{2}\right) \tag{30}$$

$$= \frac{11}{2}. \tag{31}$$

(c) Now

$$G(2) = \int_0^2 f(t)\,dt = \frac{f(0)+f(2)}{\cancel{2}} \cdot \cancel{2} = 4 + (-4) = 0, \tag{32}$$

and G, by the Fundamental Theorem of Calculus, is continuous. This means that $\lim_{x\to 2} G(x) = G(2) = 0$. Also, $\lim_{x\to 2}(x^2 - 2x) = 4 - 4 = 0$, so we may apply l'Hôpital's rule to obtain

$$\lim_{x\to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \frac{f(x)}{2x - 2}, \tag{33}$$

provided that the latter limit exists. But from the graph, we see that $\lim_{x\to 2} f(x) = -4$. Hence, the limit on the right side of (33) exists and is $-4/2 = -2$. We conclude, by l'Hôpital's rule, that

$$\lim_{x\to 2} \frac{G(x)}{x^2 - 2x} = -2. \tag{34}$$

(d) The average value, A, of the rate of change of G on the interval $[-2, 4]$ is given by

$$A = \frac{1}{4-(-2)} \int_{-2}^{4} G'(t)\,dt \tag{35}$$

$$= \frac{1}{6} \int_{-2}^{4} f(t)\,dt \tag{36}$$

$$= \frac{1}{6} \int_{-2}^{0} f(t)\,dt + \frac{1}{6} \int_{0}^{4} f(t)\,dt \tag{37}$$

$$= \frac{1}{12}\big[f(-2) + f(0)\big] \cdot 2 + \frac{1}{12}\big[f(0) + f(2)\big] \cdot 2 + \frac{1}{12}\big[f(2) + f(4)\big] \cdot 2 \tag{38}$$

$$= \frac{6+4}{6} + \frac{4-4}{6} + \frac{-4+(-2)}{6} = \frac{2}{3}. \tag{39}$$

(The trapezoidal integration is justified by the fact that f is piecewise linear and the appropriate choice of points within the interval of integration.)

(e) The function G is, by the Fundamental Theorem of Calculus, continuous on the interval $[-2,4]$ and differentiable on the interval (-2.4); moreover, $G'(x) = f(x)$ for $-2 < x < 4$. The Mean Value Theorem therefore guarantees the existence of $\xi \in (-2,4)$ such that

$$f(\xi) = G'(\xi) = \frac{G(4) - G(-2)}{4 - (-2)} = \frac{1}{6}\left[G(4) - G(-2)\right]. \tag{40}$$

But then by the definition of G

$$\begin{aligned} G'(\xi) &= \frac{1}{6}\left[\int_0^4 f(t)\,dt - \int_0^{-2} f(t)\,dt\right] && (41)\\ &= \frac{1}{6}\left[\int_{-2}^0 f(t)\,dt + \int_0^4 f(t)\,dt\right] = \frac{1}{6}\int_{-2}^4 f(t)\,dt && (42) \end{aligned}$$

According to (36), this is just A, so the answer is "Yes, the Mean Value Theorem guarantees a value ξ, $-4 < \xi < 2$, for which $G'(\xi)$ is the average rate of change of G on $[-4,2]$."

Remark: It isn't at all difficult—though it is a bit tedious—by reading the given graph, to write an explicit piecewise representation of the function f, and, thereby, of the function G.

The function f is given by

$$f(t) = \begin{cases} 3(t+4) = 3t + 12; & -4 \le f < -2 \\ 6 - (t+2) = -t + 4; & -2 \le t < 0 \\ 4 - 4t = -4t + 4; & 0 \le t < 2 \\ -4 + (t-2) = t - 6; & 2 \le t < 6. \end{cases} \tag{43}$$

Carrying out the necessary integrations, we find that

$$\tag{44}$$

$$G(x) = \begin{cases} \frac{3}{2}x^2 + 12x + 8; & -4 \le t < -2 \\ -\frac{1}{2}x^2 + 4x; & -2 \le x < 0 \\ -2x^2 + 4x; & 0 \le x < 2 \\ \frac{1}{2}x^2 - 6x + 10; & 2 \le x \le 6. \end{cases} \tag{45}$$

5. **Solution:**

(a) Beginning with the equation

$$2y^2 - 6 = y \sin x, \tag{46}$$

we differentiate both sides with respect to y, while treating y as a function of x, to see that

$$4y\frac{dy}{dx} = \sin x \cdot \frac{dy}{dx} + y\cos x, \text{ or} \tag{47}$$
$$\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}. \tag{48}$$

(b) Putting $x = 0$ and $y = \sqrt{3}$ in (48), we obtain

$$\left.\frac{dy}{dx}\right|_{(0,\sqrt{3})} = \frac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \frac{1}{4} \tag{49}$$

An equation for the line tangent to a differentiable curve $y = f(x)$ at the point (x_0, y_0) is $y = y_0 + f'(x_0)(x - x_0)$, so an equation for tangent line to this curve at $(0, \sqrt{3})$ is

$$y = \sqrt{3} + \frac{1}{4}x. \tag{50}$$

(c) The tangent line is horizontal at just those points where the derivative vanishes. From (48), we see that this can be so only when $y = 0$ or when, because $0 \le x \le \pi$, $x = \pi/2$. But if $y = 0$, then (46) requires that $-6 = 0$, which is not so. Also from (46), we see that we can have $x = \pi/2$ and $y > 0$ only when $y = \sqrt{3}$. We conclude that the only horizontal tangent line to the curve given by (46) occurs at the point with coordinates $(\pi/2, \sqrt{3})$.

(d) When y is near $\sqrt{3}$ and x is near $\pi/2$, $\sin x$ is near 1 and so substantially smaller than $4y$, which is near $4\sqrt{3}$. Thus the denominator of (48) is positive. On the other hand, $\cos x$ is positive for values of x just to the left of $\pi/2$, but negative for values of x just to the right of $\pi/2$. Because y remains positive throughout this region, the numerator of (48) changes sign from positive to negative as we pass from left to right along the curve through the point $(\pi/2, \sqrt{3})$. We conclude, that y' changes sign from positive to negative at $(\pi/2, \sqrt{3})$. By the first derivative test. f has a relative maximum at that point.

Remark: The second derivative test is certainly applicable here. But computing the requred second derivative promises to lead to a time-consuming calculation, and the first derivative test seems indicated.

6. **Solution:**

(a) Here is a plot of the solution to the intial value problem $3y' = 12 - y$ with $y(0) = 0$, shown on the slope field for the equation.

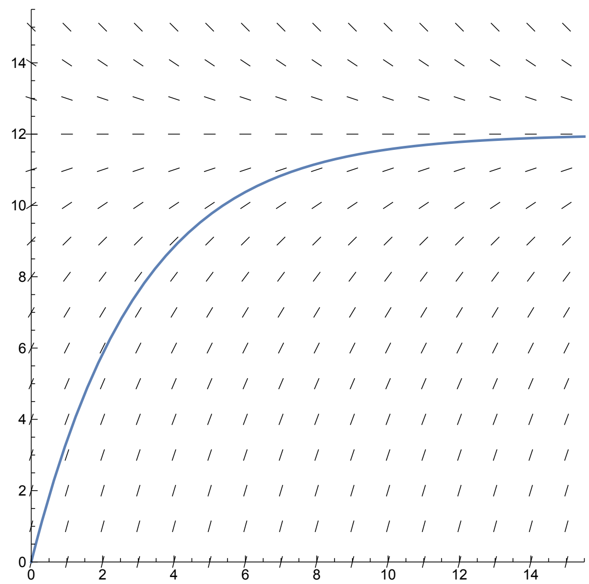

Figure 1: Solution plot for the IVP $3y' = 12 - y$; $y(0) = 0$.

(b) In the context of this problem, the statement "$\lim_{t->\infty} A(t) = 12$" means that the amount of the medication in the patient gets very close to 12 milligrams as time goes on, and that if we are willing to wait long enough, that amount will get abitrarily close to 12 milligrams.

(c) Let $y = \varphi(t)$ denote the amount of medication in the patient's bloodstream when t hours have passed since administration. We are given that, at any time τ, we have $3\varphi'(\tau) = 12 - \varphi(\tau)$. Because φ is the solution to a differential equation, it is continuous and differentiable. Because $\varphi(0) = 0$, we know that $\varphi(\tau) - 12 \neq 0$ in some interval immediately to the right of $\tau = 0$. We may, at least in that interval, write

$$\frac{\varphi'(\tau)}{12 - \varphi(\tau)} = \frac{1}{3}. \tag{51}$$

Integrating both sides with respect to τ from $\tau = 0$ to $\tau = t$, where we be sure to choose $t > 0$ sufficiently close to $\tau = 0$ that we can be sure that our division was legitimate, we write,

$$\int_0^t \frac{\varphi'(\tau)\,d\tau}{12 - \varphi(\tau)} = \frac{1}{3}\int_0^t d\tau; \tag{52}$$

The integral on the right is just $\frac{1}{3}t$. We make the substitution $u = \varphi(\tau)$ in the integral on the left. This requires that we also substitute $du = \varphi'(\tau)\,d\tau$, and that we replace the lower limit, 0, of integration with $\varphi(0) = 0$, the upper limit with $\varphi(t)$. Thus, reversing signs across the equality for convenience,

$$\int_0^y \frac{du}{u - 12} = -\frac{t}{3}, \text{ or} \tag{53}$$

$$\ln|u - 12|\Big|_0^y = -\frac{t}{3}. \tag{54}$$

This is equivalent to

$$\ln|y - 12| - \ln 12 = -\frac{t}{3}; \tag{55}$$

$$\ln|y - 12| = -\frac{t}{3} + \ln 12; \tag{56}$$

$$|y - 12| = 12e^{-t/3}. \tag{57}$$

But in the region where we have carried out the integration, y is close to zero, so $y - 12 < 0$. Hence, we may write

$$12 - y = 12e^{-t/3}. \tag{58}$$

Thus, our desired solution is given by

$$y = 12\left(1 - e^{-t/3}\right). \tag{59}$$

(d) If $\dfrac{dy}{dt} = 3 - \dfrac{y}{t+2}$ and $y = 2.5$ when $t = 1$, then

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{2.5}{1+2} > 0. \tag{60}$$

In view of this, we conclude that the amount of medication in the patient is increasing near the time $t = 1$.