Solutions to 2024 AP Calculus AB Free Response Questions

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We use the symbol "$\sim$" to mean "is approximately equal to" throughout this document.

1. (a) Reading from the given table, we find $C(3) = 85$ and $C(7) = 69$. We are to take the average rate of change for C over the interval from $t = 3$ to $t = 5$ as our approximation for $C'(5)$, and this gives

$$C'(5) \sim \frac{C(7) - C(3)}{7 - 3} = \frac{69 - 85}{4}, \text{ or} \tag{1}$$

$$C'(5) \sim -\frac{16}{4} = -4 \text{ C}^\circ/\text{min}. \tag{2}$$

(b) The data given in the table yields the left Riemann sum:

$$C(0) \cdot (3 - 0) + C(3) \cdot (7 - 3) + C(7) \cdot (12 - 7) = 100 \cdot 3 + 85 \cdot 4 + 69 \cdot 5 \quad = 985. \tag{3}$$

Thus, $\frac{1}{12}\int_0^{12} C(t)\,dt \sim \frac{985}{12}$. This means that the average temperature of the coffee during the interval between $t = 0$ minutes and $t = 12$ minutes was approximately $\frac{985}{12}$ C°.

(c) If the rate of change of the temperature of the coffee, $C'(t)$, when $12 \le t \le 20$ is given by $C'(t) = -\frac{24.55e^{0.01t}}{t}$, then the Fundamental Theorem of Calculus tells us that

$$C(20) = C(12) + \int_{12}^{20} C'(\tau)\,d\tau \tag{4}$$

$$= 55 - 24.55\int_{12}^{20} \frac{e^{0.01\tau}\,d\tau}{\tau}. \tag{5}$$

Numerical integration now leads to $C(20) \sim 40.32919$ C° ~ 40.329 C°.

(d) With $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$ and $12 \leq t \leq 20$, it is apparent that $C''(t) > 0$. (This is because the denominator of this fraction, being a square, must be positive throughout the given interval, while the only factor of the numerator that can ever be negative is $(100 - t)$—which is positive in the given interval.) Because its derivative, $C''(t)$, is positive on the interval $[12, 20]$, the quantity $C'(t)$ must be increasing on that interval. This means that the temperature of the coffee changes at an increasing rate when $12 \leq t \leq 20$.

2. In this problem velocity $v(t)$ is give by $v(t) = \ln(t^2 - 4t + 5) - 0.2t$.

   (a) If the particle is at rest when $t = t_R$, then $v(t_R) = 0$, or

   $$\ln(t^2 - 4t + 5) - 0.2t = 0. \tag{6}$$

   Solving numerically, we find that $t_R \sim 1.42561 \sim 1.425$.

   (b) Acceleration, $a(t)$, being the time derivative of velocity, we have

   $$a(t) = v'(t) \tag{7}$$
   $$= \frac{d}{dt}\left[\ln(t^2 - 4t + 5) - 0.2t\right] \tag{8}$$
   $$= \frac{2t - 4}{t^2 - 4t + 5} - 0.2, \tag{9}$$

   whence

   $$a(1.5) = \frac{2 \cdot 1.5 - 4}{(1.5)^2 - 3 \cdot (1.5) + 5} - 0.2 = -1.000. \tag{10}$$

   We know that the speed $S(t) = |v(t)|$ of the particle is never negative and satisfies

   $$[S(t)]^2 = [v(t)]^2, \tag{11}$$

   so

   $$2S(t) \cdot S'(t) = 2v(t) \cdot v'(t). \tag{12}$$

   From these observations, it follows that the sign of $S'(t)$ is always the same as the sign of the product $v(t) \cdot v'(t)$. But $v(1.5) \cdot v'(1.5) \sim (-0.768) \cdot (-1.000) > 0$, from which we see that $S'(1.5) > 0$. Because S' is continuous near $t = 1.5$, this means that $S'(t) > 0$ for t close to 1.5. We conclude that S is increasing on a small interval centered at $t = 1.5$. So speed is increasing near $t = 1.5$.

   (c) Position, $x(t)$, is related to velocity $v(t)$ by $x'(t) = v(t)$. Therefore, by the Fundamental Theorem of Calculus

   $$x(t) = x(1) + \int_1^t v(\tau)\, d\tau, \tag{13}$$

   so that

   $$x(4) = -3 + \int_1^4 \left[\ln(\tau^2 - 4\tau + 5) - 0.2\tau\right]\, d\tau. \tag{14}$$

   Numerical integration gives $x(4) \sim -2.80288 \sim -2.81$.

   (d) For the total distance traveled when $1 \leq t \leq 4$ we calculate numerically

   $$\int_1^4 \left|\ln(\tau^2 - 4\tau + 5) - 0.2\tau\right| d\tau \sim 0.95813 \sim 0.958, \tag{15}$$

   or about 0.958.

3. (a) See Figure 1.

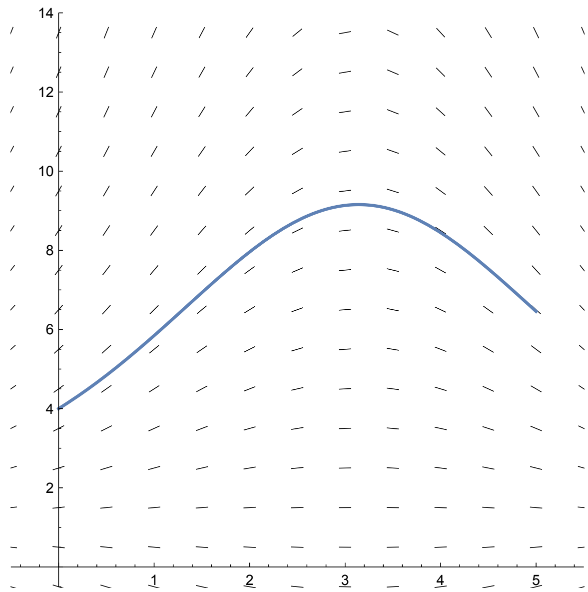

Figure 1: Solution to Problem 3(a).

(b) Critical points for a solution function $H(t)$ are to be found where $H'(t) = 0$ and where $H'(t)$ does not exist. But we are given that $H'(t)$ is defined for all t in the interval $(0, 5)$. Thus, we seek solutions to the equation

$$H'(t) = \frac{1}{2}[H(t) - 1]\cos\frac{t}{2} = 0. \tag{16}$$

We are given that $H(t) > 1$ for $0 < t < 5$. Thus, $H'(t)$ can be zero only where $\cos\frac{t}{2} = 0$ and $0 < t < 5$. We conclude that there is just one critical point, where $t = \pi$.
The second derivative $H''(t)$ is given by

$$\begin{aligned} H''(t) &= \frac{d}{dt}H'(t) && (17)\\ &= \frac{d}{dt}\left(\frac{1}{2}[H(t) - 1]\cos\frac{t}{2}\right) && (18)\\ &= \frac{1}{2}H'(t)\cos\frac{t}{2} - \frac{1}{4}[H(t) - 1]\sin\frac{t}{2} && (19) \end{aligned}$$

Now $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, and we are given that $H(\pi) > 1$ so that

$$H''(\pi) = 0 - \frac{1}{4}[H(\pi) - 1] < 0. \tag{20}$$

When its second derivative is negative, a curve is concave downward, so H is concave downward in the vicinity of the critical point $t = \pi$, and the critical point gives a local maximum for H.

(c) We are to solve the initial value problem

$$\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\frac{t}{2}; \tag{21}$$

$$H(0) = 4 \tag{22}$$

by the method of separation of variables.

From (21), we can write

$$\frac{2\,dH}{H-1} = \cos\frac{t}{2}\,dt, \tag{23}$$

whence, making use of (22),

$$\int_4^H \frac{2\,dh}{h-1} = \int_0^t \cos\frac{\tau}{2}\,d\tau. \tag{24}$$

Thus

$$\cancel{2}\ln|h-1|\Big|_4^H = \cancel{2}\sin\frac{\tau}{2}\Big|_0^t; \tag{25}$$

$$\ln\big(|H-1|\big) - \ln 3 = \sin\frac{t}{2}. \tag{26}$$

We know that $H(0) = 4$, so our solution satisfies $H(t) - 1 > 0$, at least when t is near $t = 4$. Consequently, $|H(t) - 1| = H(t) - 1$, and we may write

$$\ln(H(t) - 1) = \ln 3 + \sin\frac{t}{2}; \tag{27}$$

$$H(t) - 1 = 3e^{\sin\frac{t}{2}} \tag{28}$$

Thus, the solution, H, that we seek is given by

$$H(t) = 1 + 3e^{\sin\frac{t}{2}}. \tag{29}$$

4. (a) The part of the function f extending from the point $(0, 2)$ to $x = 7$ passes through the point $(6, -1)$ and is given to be a straight line, so when $0 \le x \le 7$,

$$f(x) = 2 - \frac{1}{2}x. \tag{30}$$

The function g is defined by

$$g(x) = \int_0^x f(t)\,dt. \tag{31}$$

Making use of the fact that we are given the value $\int_{-6}^{0} f(t)\,dt = 12$, we find that

$$g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12; \tag{32}$$

$$g(4) = \int_0^4 \left(2 - \frac{1}{2}t\right)\,dt = \left(2t - \frac{1}{4}t^2\right)\bigg|_0^4 = 4; \tag{33}$$

$$g(6) = \int_0^6 \left(2 - \frac{1}{2}t\right)\,dt = \left(2t - \frac{1}{4}t^2\right)\bigg|_0^6 = 3. \tag{34}$$

(b) The graph of $g(x) = \int_0^x f(t)\,dt$ has critical points in the interval $[0, 6]$ at those points of $(0, 6)$ where either $g'(x) = 0$ or $g'(x)$ is undefined. But the Fundamental Theorem of Calculus tells us that $g'(x) = f(x)$ throughout the domain of g. We can see from the graph of f that the only critical point for g is the single point where $f(x) = 0$—that is, at $x = 4$.

(c) If $h(x) = \int_{-6}^{x} f'(t)\,dt$, the Fundamental Theorem of Calculus, together with the fact that (reading from the graph) $f(-6) = 0.5$, tells us that

$$h(x) = f(x) - f(-6) = f(x) - 0.5. \tag{35}$$

Thus $h(6) = f(6) - 0.5$. From what is given, we see that $f(6) = -1$, so $h(6) = -1.5$.

From (35), we have $h'(x) = f'(x)$, so from (30), $h'(x) = -\frac{1}{2}$ when $0 < x < 7$. Thus, $h'(6) = f'(6) = -\frac{1}{2}$.

From the fact, already adduced, that $h'(x) = -\frac{1}{2}$ when $0 < x < 7$, we find that $h''(6) = 0$.

5. We are given, here, that

$$x^2 + 3y + 2y^2 = 48, \tag{36}$$

and, when (x, y) lies on this curve,

$$y' = -\frac{2x}{3 + 4y}. \tag{37}$$

(a) At the point $(2, 4)$, which lies on the curve (36), we have

$$y'\Big|_{(2,4)} = -\frac{4}{19}. \tag{38}$$

Thus, the equation for the line tangent to the curve (36) at $(2, 4)$ is

$$y = 4 - \frac{4}{19}(x - 2). \tag{39}$$

We can obtain the approximate value of y_0 for the point $(3, y_0)$ on the curve (36) near $(2, 4)$ by setting $x = 3$ in (39):

$$y_0 = 4 - \frac{4}{19}(3 - 2) = \frac{72}{19}. \tag{40}$$

(b) The line $y = 1$ has slope zero. From (37), we see that $y' = 0$ only when $x = 0$, so if the given line is tangent to the curve (36), its point of tangency must be $(0, 1)$. But

$$(x^2 + 3y + 2y^2)\Big|_{(0,1)} = 0^2 + 3 \cdot 1 + 2 \cdot 1^2 \neq 48, \tag{41}$$

so the coordinates of this point don't satisfy equation (36). The line $y = 1$ is not tangent to the curve (36).

(c) At the point $(\sqrt{48}, 0)$ we have

$$y'\Big|_{(\sqrt{48},0)} = -\frac{2x}{3 + 4y}\Big|_{(\sqrt{48},0)} = -\frac{2\sqrt{48}}{3} = -\frac{8\sqrt{3}}{3}, \tag{42}$$

so the line tangent to the the curve (36) has negative slope at the point $(\sqrt{48}, 0)$. That tangent line is therefore not vertical, because vertical lines have no slope.

(d) For a particle moving on the curve

$$y^3 + 2xy = 24, \tag{43}$$

we have, by implicit differentiation,

$$3y^2\frac{dy}{dt} + 2y\frac{dx}{dt} + 2x\frac{dy}{dt} = 0. \tag{44}$$

Consequently, if $\frac{dy}{dt} = -2$ when the particle is at $(4, 2)$, then

$$3(2)^2 \cdot (-2) + 2(2) \cdot \frac{dx}{dt} + 2(4)(-2) = 0, \tag{45}$$

so that

$$\frac{dx}{dt} = 10 \text{ units per second.} \tag{46}$$

6. (a) The area of the region R is given by $\int_0^2 \left[\left(x^2+2\right)-\left(x^2-2x\right)\right]\,dx$.

(b) The area of a rectangle with base B whose height is half its base is $\frac{B^2}{2}$, and the base, B, of a rectangle extending perpendicularly from the x-axis to the curve $y = g(x)$ is $B = g(x)$. The area of the solid described in the problem is therefore

$$\frac{1}{2}\int_2^5 [g(x)]^2\,dx = \frac{1}{2}\int_2^5 \left(x^4 - 4x^3 + 4x^2\right)\,dx \tag{47}$$

$$= \frac{1}{2}\left(\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3\right)\Bigg|_2^5 \tag{48}$$

$$= \frac{1}{2}\left(\frac{1}{5}\cdot 5^5 - 5^4 + \frac{4}{3}\cdot 5^3\right) - \frac{1}{2}\left(\frac{1}{5}\cdot 2^5 - 2^4 + \frac{4}{3}\cdot 2^3\right) \tag{49}$$

$$= \frac{250}{3} - \frac{8}{15} = \frac{414}{5}. \tag{50}$$

(c) $\pi\int_2^5 \left(400 - [20-(x^2-2x)]^2\right)\,dx$ gives the volume of the solid obtained by rotating the region S about the line $y = 20$.

$2\pi\int_0^{15}\left(4-\sqrt{1+y}\right)(20-y)\,dy$ is an alternate solution.