TWO SAMPLE HYPOTHESIS TEST WORKSHEET **MTH 3210**

For hypothesis tests involving two unknown population means (usually means of similar populations), we start with a *null hypothesis*, H_0, which states that the true difference of population means, $\mu_1 - \mu_2$, is equal to 0, or, equivalently, $\mu_1 = \mu_2$.

In each hypothesis test, we first select a *significance level*, α, and *alternative hypothesis*, H_0 before we collect simple random samples from each population. We denote the size of the respspective samples by n_1 and n_2. The number α represents the (small) probability that we reject the null hypothesis even though it is actually true (type I error). The means of our samples are denoted $\bar{x_1}$ and $\bar{x_2}$ and the standard deviations are denoted s_1 and s_2.

The conclusion of our hypothesis tests are always either "reject the null hypothesis " or "fail to reject the null hypothesis." We make the former conclusion if the *test statistic* falls in the *rejection region*; otherwise, we fail to reject H_0.

1. Paired t-tests for the difference between two population means (samples are *not* independent)

For these tests, we collect pairs of data from the two populations of the form $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ where the x_j's are from the first population and the y_j's are from the second population, paired according to some blocking factor. The idea is to pair the data according to a variable you may want to block the effect of, so these samples are most likely *dependent* rather than independent (more comments on this in class). We assume that the underlying population of differences has an approximately normally distributed or that the number of differences is at least 30 ($n \geq 30$). In each case the degrees of freedom is $\nu = n - 1$.

We use a *single* sample of the n differences $\{d_1, d_2, \ldots d_n\} = \{x_1 - y_1, x_2 - y_2, \ldots x_n - y_n\}$ for our sample, and denote the mean and standard deviation of this sample with $\bar{x}_d$ and s_d, respectively.

1.1. **Two-sided paired t-test.** A two-sided test uses a double interval rejection region that includes both $-\infty$ and ∞.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 \neq \mu_2$
- **Test Statistic**

$$t = \frac{\bar{d}}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

- **Critical Values** $-t_{\alpha/2,n-1}$ and $t_{\alpha/2,n-1}$
- **Rejection Region** $(-\infty, -t_{\alpha/2,n-1}]$ or $[t_{\alpha/2,n-1}, \infty)$

1.2. **The Right-sided paired t-test.** For this test, we use a single interval rejection region.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 > \mu_2$
- **Test Statistic**

$$t = \frac{\bar{d}}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

- **Critical Value** $t_{\alpha,n-1}$
- **Rejection Region** $[t_{\alpha,n-1}, \infty)$

1.3. **The Left-sided paired t-test.** For this test, we use a single interval rejection region.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 < \mu_2$
- **Test Statistic**

$$t = \frac{\bar{d}}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

- **Critical Value** $-t_{\alpha,n-1}$
- **Rejection Region** $(-\infty, -t_{\alpha,n-1}]$

2. t-tests for the difference between two population means (independent sampling)

For these tests, we make no assumption on the sizes of n_1 and n_2, but we do assume that the underlying populations are approximately normally distributed. In each case, the number of degrees of freedom we use for our t distribution is denoted by ν, where

$$\nu = \frac{(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$$

This test is sometimes called *Welch's t-test* and this equation for ν is called the *Welch-Satterthwaite equation.* We also assume that the samples are selected independently of each other.

2.1. **Two-sided t-test.** A two-sided test uses a double interval rejection region that includes both $-\infty$ and ∞.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 \neq \mu_2$
- **Test Statistic**

$$t = \frac{\bar{x_1} - \bar{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- **Critical Values** $-t_{\alpha/2,\nu}$ and $t_{\alpha/2,\nu}$
- **Rejection Region** $(-\infty, -t_{\alpha/2,\nu}]$ or $[t_{\alpha/2,\nu}, \infty)$

2.2. **The Right-sided t-test.** For this test, we use a single interval rejection region.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 > \mu_2$
- **Test Statistic**

$$t = \frac{\bar{x_1} - \bar{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- **Critical Value** $t_{\alpha,\nu}$
- **Rejection Region** $[t_{\alpha,\nu}, \infty)$

2.3. **The Left-sided t-test.** For this test, we use a single interval rejection region.

- **Null Hypothesis** $H_0 : \mu_1 = \mu_2$
- **Alternative Hypothesis** $H_A : \mu_1 < \mu_2$
- **Test Statistic**

$$t = \frac{\bar{x_1} - \bar{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- **Critical Value** $-t_{\alpha,\nu}$
- **Rejection Region** $(-\infty, -t_{\alpha,\nu}]$

3. Exercises

For these exercises, use the appropriate test based on the information given (σ or s), alternative hypothesis specified and significance level, α. You can assume that the underlying population is normally distributed.

(1) In an experiment used to test the hardness reading associated with two tips of a machine that presses a rod with a pointed tip in a manufacturing process, paired samples of hardness readings for the two tips are paired according to the metal rod that is pressed (each rod is pressed with each tip once). Ten metal rods are pressed and the table below summarizes the results:

Speciman	Tip 1	Tip 2
1	7	5
2	3	2
3	3	4
4	4	2
5	8	9
6	3	2
7	2	2
8	9	7
9	5	3
10	4	4

Test the null hypothesis for the mean difference in hardnesses given by $H_0 : \mu_d = 0$ against the alternative hypothesis $H_A : \mu_d \neq 0$ at the 10% significance level where μ_d is the true mean d difference in hardness reading over all possible specimens.

(2) Let $n_1 = 36$, $n_2 = 41$, $\bar{x}_1 = 11.6$, $\bar{x}_2 = 10.4$, $s_1 = 1.8$ and $s_2 = 2.6$. Test the hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_a : \mu_1 > \mu_2$ at the $\alpha = .05$ significance level. (Note that the Welch-Satterthwaite equation gives $\nu = 72$.)

(3) The mean tension bond strengths of two types of cement mortar (modified and unmodified) are known to be normally distributed and a cement manufacturer wishes to test the null hypothesis, $H_0 : \mu_1 = \mu_2 =$ against the alternative hypothesis $H_A : \mu_1 > \mu_2$ where μ_1 denotes the true mean tension bond strength of the modified mortar and μ_2 denotes the true mean tension bond strength of the unmodified mortar. Two simple random samples from the modified and unmodified populations of bond strengths are given by

$$\{16.85, 16.4, 17.21, 16.35, 16.52, 17, 16.96, 17.16, 16.59, 16.57\}$$

and

$$\{17.5, 17.63, 18.25, 18, 17.86, 17.75, 18.22, 17.9, 17.96, 18.15\},$$

respectively.

Use the appropriate procedure to test the null hypothesis at the 5% significance level.

(4) The article "Application of Experimental Design to the Analysis of Semiconductor Manufacturing Lines" (*Proceedings of the 1990 Winter Simulation Conference*) describes the computer simulation of a rather complex semiconductor manufacturing line. One purpose of the study was to examine the sensitivity of the cycle time (time from beginning of the manufacture completion) to the number of operators stationed at one point along the line.

The model was run separately 16 times under each of two experimental conditions: first with just one operator at the station, and then with two. The summary statistics for the two samples of cycle times are:

$$\begin{aligned} n_1 &= 16 \\ n_2 &= 16 \\ \bar{x}_1 &= 372.6 \\ \bar{x}_2 &= 374.8 \\ s_1 &= 7.8 \\ s_2 &= 7.3 \\ \nu &= 30 \end{aligned}$$

Carry out the appropriate two-sided hypothesis test using a level of significance $\alpha = 0.10$.

State your conclusion in terms of the real world problem.