AP Calculus 2013 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

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1 Problem 1

1.1 Part a

If

$$G(t) = 90 + 45\cos\frac{t^2}{18}, \tag{1}$$

then

$$G'(t) = -5t\sin\frac{t^2}{18}. \tag{2}$$

Thus

$$G'(5) = -25\sin\frac{25}{18} \text{ tons per hour per hour.} \tag{3}$$

This means that the unprocessed gravel arrival rate is decreasing at time $t = 5$ at the rate of $25\sin(25/18)$ tons per hour every hour. This is about -24.58751 tons per hour per hour.

Note: The word *decreasing* is usually defined only over intervals, and not at points. It might therefore be better to say that because $G'(5) < 0$ and G is a continuous function, G′(t) must be negative over an interval centered at $t = 5$. Thus, the amount of unprocessed gravel at the plane is decreasing over some interval centered at $t = 5$.

1.2 Part b

The total amount, in tons, of unprocessed gravel that arrives at the plant during the hours of operation is, in tons,

$$\int_0^8 G(t)\,dt.$$

Integrating numerically, we find that this is about 825.55109 tons.

1.3 Part c

The rate at which the plants unprocessed gravel changes is

$$G(t) - 100 = 45\cos\frac{t^2}{18} - 10, \tag{4}$$

and at time $t = 5$, this is

$$G(5) - 100 = 45\cos\frac{25}{18} - 10 \sim -1.85924 \text{ tons per hour per hour.} \tag{5}$$

1.4 Part d

The amount, in tons, $A(t)$ of unprocessed gravel at the plant at time t during the hours of operation on this workday is given by

$$A(t) = 500 + \int_0^t \left[45\cos\frac{\tau^2}{18} - 10\right] d\tau. \tag{6}$$

By the Fundamental Theorem of Calculus,

$$A'(t) = 45\cos\frac{t^2}{18} - 10. \tag{7}$$

We note that $A'_+(0) = 35 > 0$, while $A'_-(8) \sim -51.19903 < 0$, so A can have a maximum for $[0,8]$ only interior to the interval. By the Extreme Value Theorem, there must be a maximum, and we know that it can happen only at a point where $G(t) - 100 = A'(t) = 0$. Solving numerically, we find the only such point to be $t \sim 4.92348$. Taking t to have this value and integrating the expression in (6) numerically, we find that the maximum amount of unprocessed gravel at the plant during the hours of operation on this work day is approximately 635.37612 tons.

2 Problem 2

2.1 Part a

The intersection points for these two curves are where

$$4 - 2\sin\theta = 3, \text{ or} \tag{8}$$

$$\sin\theta = \frac{1}{2}. \tag{9}$$

These solutions are $\theta = \pi/6$ and $\theta = 5\pi/6$.

Thus, the area of the region S that lies inside both of the curves $r = 3$ and $r = 4 - \sin\theta$ is

$$\frac{1}{2}\int_{\pi/6}^{5\pi 6} (4 - 2\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{5\pi/6}^{13\pi/6} 3^2 \, d\theta. \tag{10}$$

Integrating numerically, we obtain 24.70873 as the approximate volume.

Note: Numeric integration saves time. However the integrals are both elementary.

In fact

$$\int (4 - 2\sin\theta)^2 \, d\theta = \int \left[16 - 16\sin\theta + 4\sin^2\theta\right] \, d\theta \tag{11}$$

$$= 16\theta + 16\cos\theta + 2\int [1 - \cos 2\theta] \, d\theta \tag{12}$$

$$= 18\theta + 16\cos\theta - \sin 2\theta + C. \tag{13}$$

What remains of the calculation is left to the interested reader.

2.2 Part b

We have $x(t) = r(t)\cos\theta(t)$. we want to find the value of t in the interval $[1, 2]$ for which $x(t) = -1$ with $\theta(t) = t^2$, so we must solve the equation $-1 = (4 - 2\sin t^2)\cos t^2$.

A plot shows that there is just one solution to this equation in the interval, and numerical solution gives that solution as $t \sim 1.42798$.

2.3 Part c

The position vector for the particle described in Part b is

$$\mathbf{r}(t) = \langle (4 - 2\sin t^2)\cos t^2, (4 - 2\sin t^2)\sin t^2 \rangle. \tag{14}$$

so the velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -4t\cos^2 t^2 - 2t(4 - 2\sin t^2)\sin t^2, 2t\cos t^2(4 - 2\sin t^2) - 4t\cos t^2 \sin t^2 \rangle. \tag{15}$$

When $t = 1.5$, this gives

$$\mathbf{v}(1.5) \sim \langle -8.07210, -1.67290 \rangle. \tag{16}$$

3 Problem 3

3.1 Part a

The value of $C'(3.5)$ is approximately

$$C'(3.5) \sim \frac{C(4) - C(3)}{4 - 3} = \frac{12.8 - 11.2}{1} = 1.6 \text{ ounces per minute.} \tag{17}$$

3.2 Part b

We have

$$\frac{C(4) - C(2)}{4 - 2} = \frac{12.8 - 8.8}{2} = 2. \tag{18}$$

It is given that C is a differentiable function, presumably on $[0, 6]$ though this is somewhat unclear. So C is continuous on $[2, 4]$ and differentiable on $(2, 4)$. C thus satisfies the hypotheses of the Mean Value Theorem on the interval $[2, 4]$. Taking (18) into account, we see that there must be a value t_0 in $(2, 4)$ such that

$$C'(t_0) = 2. \tag{19}$$

3.3 Part c

The midpoint sum with three subintervals of equal length indicated by the data in the table for $\frac{1}{6}\int_0^6 C(\tau)\,d\tau$ is

$$\frac{1}{6}\left[5.3\cdot(2-0)+11.2\cdot(4-2)+13.8\cdot(6-4)\right]=10.1. \tag{20}$$

In the context of the problem, this means that the average amount of coffee in the cup over the time period $0 \le t \le 6$ is about 10.1 ounces.

3.4 Part d

If $B(t)$, the amount of coffee, in ounces, in the cup at time t, is given by $B(t) = 16-16e^{-04.t}$, then $B'(t) = 6.4e^{-0.4t}$, so that $B'(5) = 6.4e^{-2} \sim 0.86614$ ounces per minute. This is the rate at which the amount of coffee in the cup is changing when $t = 5$.

4 Problem 4

4.1 Part a

By the Fundamental Theorem of Calculus, we have

$$f(x) = f(8) + \int_8^x f'(t)\,dt \tag{21}$$

$$= 4 + \int_8^x f'(t)\,dt \tag{22}$$

$$= 4 - \int_x^8 f'(t)\,dt. \tag{23}$$

The function f can have a local minimum on the open interval $(0, 8)$ only at a point where f' increases through zero as x increases through that point. According to the graph, the only such point is at $x = 6$.

4.2 Part b

The required absolute minimum must occur either at an endpoint or at a critical point which is also a local minimum. We are given that $f(8) = 4$. At the local minimum found in Part a, above, we have

$$f(6) = 4 - \int_6^8 f'(t)\,dt = -3. \tag{24}$$

At the left-hand endpoint we have

$$\begin{aligned} f(0) &= 4 - \int_0^8 f'(t)\,dt & (25)\\ &= 4 - \int_0^1 f'(t)\,dt - \int_1^4 f'(t)\,dt - \int_4^6 f'(t)\,dt - \int_6^8 f'(t)\,dt & (26)\\ &= 4 - 2 - 6 + 3 - 7 = -8. & (27) \end{aligned}$$

It follows that the absolute minimum for f on $[0, 8]$ is $f(0) = -8$.

4.3 Part c

A function, F, is concave down and increasing on any interval where $F'(x)$ is positive (except, perhaps, for some isolated zeros) and decreasing. We therefore seek open intervals where the graph of f', which is given, is above the x-axis and has tangent lines of negative slope. There are two such intervals: $(0, 1)$ and $(3, 4)$.

Note: F is increasing on the closures of both these intervals, because a continuous function that is increasing on some open interval must be increasing on the closure of that interval. Whether a similar statement is true of downward (upward) concavity depends on which of several common definitions one chooses to employ.

4.4 Part d

If $g(x) = [f(x)]^3$, then $g'(x) = 3[f(x)]^2 f'(x)$. From the graph, $f'(3) = 4$. Hence,

$$g'(3) = 3\,[f(3)]^2\, f'(3) = 3 \cdot \left(-\frac{5}{2}\right)^2 \cdot 4 = 75. \tag{28}$$

Thus, the slope of the line tangent to the curve $y = g(x)$ at $x = 3$ is $g'(3) = 75$.

5 Problem 5

5.1 Part a

Because f, as the solution to the differential equation

$$\frac{dy}{dx} = y^2(2x+2), \tag{29}$$

is continuous on its domain and $f(0) = -1$ we know that $\lim_{x\to 0} f(x) = -1$. Hence, $\lim_{x\to 0}[f(x)+1] = 0$. In addition, $\lim_{x\to 0} \sin x = 0$. We may therefore attempt to evaluate the limit by using l'Hôpital's rule. We have

$$\frac{\frac{d}{dx}[f(x)+1]}{\frac{d}{dx}\sin x} = \frac{f'(x)}{\cos x}. \tag{30}$$

But $f'(x) = [f(x)]^2(2x+2)$, by (29), so

$$\frac{\frac{d}{dx}[f(x)+1]}{\frac{d}{dx}\sin x} = \frac{[f(x)]^2(2x+2)}{\cos x}. \tag{31}$$

But $\lim_{x\to 0} y^2 = [\lim_{x\to 0}[f(x)]^2 = (-1)^2 = 1$, so

$$\lim_{x\to 0} \frac{\frac{d}{dx}[f(x)+1]}{\frac{d}{dx}\sin x} = \frac{\lim_{x\to 0}\left\{[f(x)]^2(2x+2)\right\}}{\lim_{x\to 0}\cos x} = 2, \tag{32}$$

and it follows from l'Hôpital's rule that

$$\lim_{x\to 0} \frac{f(x)+1}{\sin x} = 2. \tag{33}$$

5.2 Part b

Euler's method with step-size h and initial condition (x_0, y_0) gives

$$x_n = x_0 + nh, \tag{34}$$
$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h, \tag{35}$$

for $n > 0$.

In this instance, we desire y_2, where $h = 1/4$, $x_0 = 0$, $y_0 = -1$, and $f(x, y) = y^2(2x + 2)$. We have

$$x_1 = 0 + \frac{1}{4} = \frac{1}{4}; \tag{36}$$

$$y_1 = (-1) + 2 \cdot \frac{1}{4} = -\frac{1}{2}; \tag{37}$$

$$x_2 = 0 + 2 \cdot \frac{1}{4} = \frac{1}{2}; \tag{38}$$

$$y_2 = -\frac{1}{2} + \frac{5}{8} \cdot \frac{1}{4} = -\frac{11}{32}. \tag{39}$$

It follows that the approximate value of $f(1/2)$ is $-11/32$.

5.3 Part c

If $y = f(x)$ is the solution to the initial value problem of Part b of this problem, then $f(x) < 0$ on some interval I centered at $x = 0$, because $f(0) = -1$ and, as the solution of an initial value problem, f must be continuous near $x = 1$. In the interval I, we can therefore rewrite the equation $f'(x) = y^2(2x + 2)$ as

$$\frac{f'(x)}{[f(x)]^2} = 2x + 2. \tag{40}$$

Thus, for any x in I, we have

$$\int_0^x \frac{f'(\xi)}{[f(\xi)]^2}\, d\xi = \int_0^x (2\xi + 2)\, d\xi, \text{ or} \tag{41}$$

$$-\frac{1}{f(\xi)}\bigg|_0^x = \left(\xi^2 + 2\xi\right)\bigg|_0^x. \tag{42}$$

Substituting as indicated and solving, we find that $f(x) = -(x + 1)^{-2}$.

Note: We can use this solution to solve the limit problem of Part a:

$$\lim_{x \to 0} \frac{f(x) + 1}{\sin x} = \lim_{x \to 0} \frac{-(x + 1)^{-2} + 1}{\sin x} \tag{43}$$

$$= \lim_{x \to 0} \frac{-1 + (x + 1)^2}{(x + 1)^2 \sin x} \tag{44}$$

$$= \lim_{x \to 0} \frac{x^2 + 2x}{(x + 1)^2 \sin x} \tag{45}$$

$$= \lim_{x \to 0} \left[\frac{x + 2}{(x + 1)^2} \cdot \frac{x}{\sin x}\right] \tag{46}$$

$$= 2 \cdot 1 = 2. \tag{47}$$

6 Problem 6

6.1 Part a

We have $P_1(x) = f(0) + f'(0)x$, so, $f(0) = -4$ being given,

$$-3 = P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2}. \tag{48}$$

Thus, $f'(0) = 2 \cdot (-3 + 4) = 2$.

6.2 Part b

$$\begin{aligned} P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 && (49) \\ &= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3. && (50) \end{aligned}$$

6.3 Part c

We note that if $h'(x) = f(2x)$, then $h''(x) = 2f'(2x)$ and $h'''(x) = 4f''(2)$. From the given equality $h(0) = 7$, we have $h'(0) = f(0) = -4$ (given in Part a, above), $h''(0) = 2f'(0) = 2 \cdot 2 = 4$ (from Part a, above), and $h'''(0) = 4f''(0) = 4(-2/3) = -8/3$ (the value of $f''(0)$ being from Part b, above). It follows that the third-degree Taylor polyomial for h about $x = 0$ is

$$h(0) + h'(0)x + \frac{h''(0)}{2}x^2 + \frac{h'''(0)}{6}x^3 = 7 - 4x + 2x^2 - \frac{4}{9}x^3. \tag{51}$$