Galois Action on the Principal Block and Cyclic Sylow Subgroups

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Abstract. We characterize finite groups $G$ having a cyclic Sylow $p$-subgroup in terms of the action of a specific Galois automorphism on the principal $p$-block of $G$, for $p = 2, 3$. We show that the analog statement for blocks with arbitrary defect group would follow from the blockwise Galois–McKay conjecture.

Introduction

One of the most prevalent questions in the representation theory of finite groups is to determine what relationships hold between the set $\text{Irr}(G)$ of irreducible complex characters of a finite group $G$ and its local structure, such as the structure of a Sylow $p$-subgroup $P$ of $G$. There is, of course, the more sophisticated question of relating the set $\text{Irr}(B)$ of irreducible characters belonging to a given Brauer $p$-block $B$ of $G$ with the structure of a defect group $D$ of $B$.

In [NT19], G. Navarro and P. H. Tiep conjecture that for a prime $p$, one can determine the exponent of the abelianization of $P$ in terms of the action of certain Galois automorphisms on $\text{Irr}(G)$. To be more precise, for a fixed prime $p$ and an integer $e \geq 1$, let $\sigma_e \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) = \mathcal{G}$ be such that $\sigma_e$ fixes $p'$-roots of unity and sends any root of unity of order a power of $p$ to its $(p^e + 1)$st power. In [NT19] it is proven that the exponent of $P/P'$ is less than or equal to $p^e$ whenever all of the irreducible characters of $p'$-degree of $G$ are $\sigma_e$-fixed, and the converse is reduced to a question on finite simple groups. (Thanks to [Mal19] we know that the converse holds for $p = 2$.)

In the present work, we show that one can determine whether $P$ is cyclic only in terms of the action of a specific $\sigma_e$ on $\text{Irr}(B_0)$, where $B_0$ is the principal $p$-block of $G$. This is the main result of our paper.
Theorem A. Let $G$ be a finite group of order divisible by $p$, where $p \in \{2, 3\}$. Let $P \in \text{Syl}_p(G)$ and let $B_0$ be the principal $p$-block of $G$. Then

$$|\text{Irr}_{p'}(B_0)^{\sigma_1}| = p$$

if, and only if, $P$ is cyclic,

where $\text{Irr}_{p'}(B_0)^{\sigma_1}$ is the set of irreducible characters in $B_0$ with degree relatively prime to $p$ that are fixed under the action of $\sigma_1$.

We remark that the conclusion of Theorem A does not hold for $p > 3$, as can be seen by considering the dihedral group $D_{2p}$, which satisfies $|\text{Irr}(B_0(D_{2p}))^{\sigma_1}| < p$.

With the definition above, $\sigma_\xi$ is an element of the subgroup $H \leq G$ consisting of all $\sigma \in G$ for which there exists some integer $f$ such that $\sigma(\xi) = \xi^f$ whenever $\xi$ is a root of unity of order not divisible by $p$. Navarro predicted in [Nav04, Conjecture A] the existence of bijections for the McKay conjecture commuting with the action of $H$ on characters. This is the celebrated Galois–McKay conjecture, which has been recently reduced to a question on finite simple groups in [NSV19]. The Galois–McKay conjecture admits a blockwise version [Nav04, Conjecture B], sometimes known as the Alperin–Galois–McKay conjecture (as it can be also seen as a refined version of the celebrated Alperin–McKay conjecture), which remains unreduced at the present moment. In this context, it is natural to wonder the extent to which Theorem A holds for arbitrary blocks. We propose the following.

Conjecture B. Let $p \in \{2, 3\}$. Let $G$ be a finite group and let $B$ be a $p$-block of $G$ with nontrivial defect group $D$. Then

$$|\text{Irr}_0(B)^{\sigma_1}| = p$$

if, and only if, $D$ is cyclic,

where $\text{Irr}_0(B)^{\sigma_1}$ is the set of height zero irreducible characters in the block $B$ that are fixed under the action of $\sigma_1$.

We prove that Conjecture B follows from the Alperin–Galois–McKay conjecture. In this sense, in addition to providing a method to determine whether $P$ is cyclic from the set $\text{Irr}(B_0)$, Theorem A provides more evidence of the elusive Alperin–Galois–McKay conjecture. Since the latter holds whenever $D$ is cyclic, by work of Navarro in [Nav04], it follows that the “if” direction of Conjecture B (and of Theorem A) holds.

This paper is structured as follows. In Section 1 we prove that Conjecture B follows from the Alperin–Galois–McKay conjecture. To do so, we study the action of $\sigma_1$ on the irreducible characters of blocks with normal defect group. The rest of the paper is devoted to proving Theorem A. In Section 2, we reduce Theorem A to statements on finite simple groups, and in Section 3, we prove those statements.

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1. Blocks with normal defect group

The aim of this section is to prove that Conjecture B follows from the Alperin–Galois–McKay conjecture, stated below.

For a fixed prime $p$, consider the set $\text{Bl}(G)$ of Brauer ($p$-)blocks of $G$ as in [Nav98], so that $\text{Bl}(G)$ is a partition of $\text{Irr}(G) \cup \text{IBr}(G)$ (recall that $p$-Brauer characters are defined on $p$-regular elements of $G$). Write $\text{Irr}(B) = B \cap \text{Irr}(G)$ and $\text{IBr}(B) = B \cap \text{IBr}(G)$ for any $B \in \text{Bl}(G)$. Every block $B$ has associated a uniquely defined conjugacy class of $p$-subgroups of $G$, namely its defect groups. Given a block $B$ of $G$ with defect group $D$, we write $B \in \text{Bl}(G)$ and let $b \in \text{Bl}(N_G(D)|D)$ denote its Brauer first main correspondent. Finally, $\chi \in \text{Irr}(B)$ has height zero in $B$ if $\chi(1)_p = |G : D|_p$, and we write $\text{Irr}_0(B)$ to denote the subset of height zero characters in $\text{Irr}(B)$.

Assuming the notation of the Introduction, we have that the group $G$ acts on $\text{Irr}(B)$ for every $B \in \text{Bl}(G)$ by [Nav98, Theorem 3.19]. The group $H$ further acts on the set $\text{Bl}(G)$ by [Nav04, Theorem 2.1]. While the action of $G$ on characters is not natural enough in global-local contexts, Navarro conjectured the following in [Nav04].

**Conjecture** (Alperin–Galois–McKay conjecture). Let $B \in \text{Bl}(G)$ and let $b \in \text{Bl}(N_G(D)|D)$ be its Brauer first main correspondent. If $\sigma \in H$, then

$$|\text{Irr}_0(B)^\sigma| = |\text{Irr}_0(b)^\sigma|.$$ 

Here we are only concerned with the action of a specific element of $H$, namely $\sigma_1$. Recall that $\sigma_1 \in H$ fixes $p'$-roots of unity and sends any root of unity of order a power of $p$ to its $(p + 1)$st power. If $G$ is a finite group of order dividing some integer $n$, then by elementary number theory, the restriction $\omega$ of $\sigma_1$ to the $n$th cyclotomic field $\mathbb{Q}(\xi_n)$ has order a power of $p$, and $\omega$ acts as $\sigma_1$ on the ordinary characters of every subgroup of $G$. Abusing notation, we will also write $\sigma_1$ for any such restriction. In particular, $\sigma_1$ fixes the elements of $\text{IBr}(G)$, and hence acts trivially on $\text{Bl}(G)$. (Note that in general $G$ does not act on $\text{IBr}(G)$, but $H$ does by Theorem 2.1 of [Nav04].)

We find it worth mentioning that a version of the conjecture above, but restricted to the action of Galois automorphisms of order a power of $p$ and fixing $p'$-roots of unity, first appeared as Conjecture D in [IN02] (a paper that can be considered as the genesis of the remarkable Galois–McKay and Alperin–Galois–McKay conjectures).
Returning to our goal, in order to prove that Conjecture B follows from the Alperin–Galois–McKay conjecture, we need to study blocks with a normal defect group. We follow the notation in Chapter 9 of [Nav98]. Let $B \in \text{Bl}(G|D)$ and assume that $D \lhd G$. Write $C = C_G(D)$. We will denote by $b \in \text{Bl}(CD|D)$ a root of $B$, and we will let $\theta \in \text{Irr}(b)$ be the canonical character associated with $B$, which is unique up to $G$-conjugacy (see [Nav98, Theorem 9.12] and the subsequent discussion). Recall that $D \subseteq \ker(\theta)$ and $\theta$ has $p$-defect zero when viewed as a character of $CD/D$ (that is, $\theta(1)_p = |CD : D|_p$), the stabilizer of the block $b$ is $G_b = G_{\theta}$, and the inertial index $|G_{\theta} : CD|$ is not divisible by $p$. In this situation, the set $\text{Irr}(b)$ is parametrized by the set $\text{Irr}(D)$, in the following sense. We may write $\text{Irr}(b) = \{ \theta_\lambda \mid \lambda \in \text{Irr}(D) \}$, where the irreducible characters $\theta_\lambda \in \text{Irr}(CD)$ are defined for $x \in CD$ as follows: $\theta_\lambda(x) = \chi(x_1)\theta(x_1)$ if $x_1 \in D$ and $\theta_\lambda(x) = 0$ otherwise. Notice that

$$G_{\theta_\lambda} = G_{\theta} \cap G_{\lambda}.$$  

Indeed, let $g \in G_{\theta_\lambda}$, so $g \in G_b = G_{\theta}$. We need to show that $g \in G_{\lambda}$. If $x \in D$, then $\lambda(x)\theta(1) = \theta_\lambda(x) = (\theta_\lambda)^g(x) = \theta_\lambda(gxg^{-1}) = \lambda^g(x)\theta(1)$. Hence $\lambda^g = \lambda$ and $g \in G_{\lambda} \cap G_{\theta}$. The converse is clear.

Let $c \in \text{Bl}(G_b|D)$ be the Fong-Reynolds correspondent of $b$ and $B$ as in [Nav98, Theorem 9.14]. Then the induction map $\text{Irr}(c) \to \text{Irr}(B)$ defines a height-preserving bijection. By [Nav98, Theorems 9.21 and 9.22] $c = b^{G_b}$ is the only block of $G_b$ that covers $b$, and one can see that

$$(1) \quad \text{Irr}(B) = \bigcup_{\lambda \in \text{Irr}(D)} \text{Irr}(G|\theta_\lambda).$$

It is not difficult to see that height zero characters of $B$ further lie over characters parametrized by linear characters of $D$, so that

$$(2) \quad \text{Irr}_0(B) = \bigcup_{\lambda \in \text{Irr}(D/D')} \text{Irr}(G|\theta_\lambda).$$

We start by explicitly describing the set $\text{Irr}_0(B)^{\sigma_1}$ when the defect group of $B$ is normal. Since [NT19, Lemma 5.1] is key for such a description and will often be used in future sections of this work, we state it below for the reader’s convenience.

**Lemma 1.1.** Let $G$ be a finite group and $\xi|G|$ a primitive $|G|$th root of unity. Suppose that $\sigma \in \text{Gal}(\mathbb{Q}(\xi|G|)/\mathbb{Q})$ has order a power of $p$ and fixes $p'$-roots of unity. Suppose that $N \lhd G$ has $p'$-index, and let $\theta \in \text{Irr}(N)$ be $\sigma$-invariant. Then every $\chi \in \text{Irr}(G|\theta)$ is $\sigma$-invariant.

**Proof.** This is [NT19, Lemma 5.1]. \hfill \Box

We also need the following technical lemma.

**Lemma 1.2.** Let $G$ be a finite group and let $p$ be a prime. Suppose that $B$ is a block of $G$ with normal defect group $D$. Let $b$ be a root of $B$ with canonical character
Write \( A = \langle \sigma_1 \rangle \leq \text{Gal}(\mathbb{Q}(\xi) / \mathbb{Q}) \). If \( \lambda \) is a linear character of \( D \), then let \( G_{\theta, \lambda} = \{ g \in G \mid (\theta \lambda)^g = (\theta \lambda)^a \text{ for some } a \in A \} \). With this definition
\[
G_{\theta, \lambda} = G_{\theta, \lambda} = G_{\theta} \cap G_{\lambda}.
\]

**Proof.** Write \( C = C_G(D) \). Recall that \( b \) is a block of \( CD \) of defect \( D \) and \( \theta \in \text{Irr}(CD) \) has defect zero as a character of \( CD/D \). Note that \( \theta \) is \( A \)-fixed since \( b^\theta = b \) for every \( a \in A \). Let \( g \in G_{\theta, \lambda} \). We start by proving that \( g \in G_{\theta} \). Since \( \theta \) is \( A \)-fixed, by the definition of \( \theta \), we have \( (\theta \lambda)^g = (\theta \lambda)^a = \theta \lambda^a \) for some \( a \in A \). Evaluating on \( D \) we see that
\[
\theta(1)\lambda^a(x) = \theta_\lambda^a(x) = \theta(x) = \theta(1)\lambda^a(x),
\]
for every \( x \in D \). Hence \( \lambda^a = \lambda^a \). Let \( x \in CD \) be such that \( xD \in (CD/D)^0 \), the set of \( p \)-regular elements of \( CD/D \), and notice that \( x_p \in D \). (Otherwise \( \theta(x) = 0 \).) Then
\[
\lambda^a(x_p)\theta^a(x_p) = \theta^a(x) = \theta^a(x_p) = \lambda^a(x_p)\theta(x_p) = \lambda^a(x_p)\theta(x_p).
\]
This implies \( \theta^a(x_p) = \theta(x) \). Since \( xD = x_pD \), then \( \theta^a = \theta \) and \( g \in G_{\theta} \).

Next we prove that \( g \in G_{\lambda} \). We know that \( \lambda^a = \lambda^a \) for some \( a \in A \), and that \( g \in G_{\lambda} \). Since \( G_{\lambda}/CD \) is a \( p \)-group, then \( \lambda^m = \lambda \) for some integer \( m \) relatively prime to \( p \). In particular, \( \lambda^m = \lambda \) and the order of \( a \) as an element of \( \text{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) = \text{Gal}(\mathbb{Q}(\xi_{\phi(\lambda)})/\mathbb{Q}) \) divides \( m \), which forces \( a = 1 \) and \( \lambda^a = \lambda \), as wanted. \( \square \)

We may now describe the set \( \text{Irr}_0(B)^{\sigma_1} \) in the case that a defect group of \( B \) is normal.

**Theorem 1.3.** Let \( G \) be a finite group and let \( p \) be a prime. Suppose that \( B \) is a block of \( G \) with a normal defect group \( D \). Let \( b \) be a root of \( B \) with canonical character \( \theta \). Then
\[
\text{Irr}_0(B)^{\sigma_1} = \bigcup_{\lambda \in \text{Irr}(D/\Phi(D))} \text{Irr}(G[\theta_\lambda]),
\]
where \( \Phi(D) \) is the Frattini subgroup of \( D \). Moreover, if \( c \in \text{Bl}(G_b|D) \) is the Fong-Reynolds correspondent of \( B \), then
\[
|\text{Irr}_0(B)^{\sigma_1}| = |\text{Irr}_0(c)^{\sigma_1}|.
\]

**Proof.** First notice that as a \( p \)-group, \( D \) has a unique block, the principal one, and \( \text{Irr}_0(B_0(D)) = \text{Irr}_0(D) = \text{Irr}(D/\Phi(D)) \). Then \( \text{Irr}_0(D)^{\sigma_1} = \text{Irr}(D/\Phi(D)) \). Since \( D/\Phi(D) \) is \( p \)-elementary abelian, one inclusion is straight-forward. To see that \( \text{Irr}_p(D)^{\sigma_1} \subseteq \text{Irr}(D/\Phi(D)) \) notice that if \( \lambda \in \text{Irr}(D/\Phi(D)) \) is \( \sigma_1 \)-fixed, then \( \lambda^{\sigma_1} = \lambda^{p+1} = \lambda \), hence \( |D/\ker(\lambda)| \leq p \), implying \( \Phi(D) \subseteq \ker(\lambda) \).

Write \( A = \langle \sigma_1 \rangle \) and let \( G_{\theta, \lambda} \) be as in Lemma 1.2. By Equation (2), we know that
\[
\text{Irr}_0(B) = \bigcup_{\lambda \in \text{Irr}(D/\Phi(D))} \text{Irr}(G[\theta_\lambda]).
\]
If \( \chi \in \text{Irr}_0(B)^{\sigma_1} \) lies over \( \theta_\lambda \), then \( (\theta_\lambda)^{\sigma_1} = (\theta_\lambda)^g \), for some \( g \in G \). In particular, \( g \in G_{\theta_\lambda} = G_{\theta} \cap G_\lambda \) by Lemma 1.2. Then \( \lambda^{\sigma_1} = \lambda^g = \lambda \). Hence \( \Phi(D) \subseteq \ker(\lambda) \) and \( \lambda \in \text{Irr}(D/\Phi(D)) \).

Conversely, let \( \chi \in \text{Irr}(G|\theta_\lambda) \), where \( \lambda \in \text{Irr}(D/\Phi(D)) \). Then \( \lambda^{\sigma_1} = \lambda \). As \( b^{\sigma_1} = b \), we see \( \sigma_1 \) fixes \( \theta \) too. Then \( (\theta_\lambda)^{\sigma_1} = \theta_\lambda \). Let \( \psi \in \text{Irr}(G_{\theta_\lambda}) \) be the Clifford correspondent of \( \chi \) over \( \theta_\lambda \). Since \( G_{\theta_\lambda} \subseteq G_{\theta} \), we know that \( p \) does not divide the order of \( G_{\theta_\lambda}/CD \).

By Lemma 1.1, \( \psi \) is \( \sigma_1 \)-invariant and so is \( \chi \).

To prove the last part of the statement, recall that the Fong-Reynolds correspondence states that the induction map \( \psi \mapsto \psi^G \) provides a bijection \( \text{Irr}_0(c) \rightarrow \text{Irr}_0(B) \).

In particular, \( |\text{Irr}_0(c)^{\sigma_1}| \leq |\text{Irr}_0(B)^{\sigma_1}| \). Now let \( \chi \in \text{Irr}_0(B)^{\sigma_1} \) lie over \( \theta_\lambda \), for some \( \lambda \in \text{Irr}(D/\Phi(D)) \) by the first part of this proof. Then \( (\theta_\lambda)^{\sigma_1} = (\theta_\lambda)^g \) for some \( g \in G \). In particular, \( g \in G_{\theta_\lambda} \). Since \( G_{\theta_\lambda} = G_{\theta} \) by Lemma 1.2, \( \theta_\lambda \) is \( \sigma_1 \)-fixed. Let \( \xi \in \text{Irr}(G_{\theta_\lambda}|\theta_\lambda) \) be the Clifford correspondent of \( \chi \). Since both \( \chi \) and \( \theta_\lambda \) are \( \sigma_1 \)-fixed then so is \( \xi \). We have that \( \xi^{G_{\theta_\lambda}} \) is the Fong-Reynolds correspondent of \( \chi \) by the transitivity of block induction (see [Nav98, Problem 4.2]), which is \( \sigma_1 \)-fixed. \( \square \)

The Alperin–Galois–McKay conjecture holds for blocks with cyclic defect groups by [Nav04, Theorem 3.4]. We obtain the following as a consequence of this fact.

**Theorem 1.4.** Let \( G \) be a finite group and let \( B \) be a block of \( G \) with cyclic defect group \( D \). Then

\[ 1 \leq |\text{Irr}_0(B)^{\sigma_1}| \leq p. \]

The set \( \text{Irr}_0(B)^{\sigma_1} \) has minimal size 1 if, and only if, \( D \) is trivial. Furthermore, if \( p \in \{2, 3\} \) and \( D \) is nontrivial, then

\[ |\text{Irr}_0(B)^{\sigma_1}| = p. \]

**Proof.** By [Nav04, Theorem 3.4], we may assume that \( D \unlhd G \). Write \( C = C_G(D) \supseteq D \).

Let \( b \in \text{Bl}(C|D) \) be a root of \( B \) with canonical character \( \theta \). By Theorem 1.3, we may assume that \( \theta \) is \( G \)-invariant (in particular, \( G/C \) is a \( p' \)-group) and

\[ \text{Irr}_0(B)^{\sigma_1} = \bigcup_{\lambda \in \text{Irr}(D/\Phi(D))} \text{Irr}(G|\theta_\lambda) \subseteq \text{Irr}(G|\Phi(D)). \]

Write \( \overline{G} = G/\Phi(D) \) and use the bar convention. Let \( \overline{F} = C_{\overline{G}}(\overline{D}) \), where \( \Phi(D) \subseteq F \leq G \). We claim that \( F = C \). Clearly \( C \subseteq F \). Note that \( \overline{F} \) acts trivially on \( \overline{D} \) and coprimely on \( \overline{D} \). By [Isa08, Theorem 3.29] we have that \( \overline{F} \) acts trivially on \( \overline{D} \) as well. Thus \( F = C \) as claimed.

Notice that since \( D \) is cyclic and \( G/C \) is a \( p' \)-group, then \( G/C \) is isomorphic to a subgroup of \( C_{p-1} \). Say \( |G/C| = m \) and let \( \{\lambda_i\}_{i=1}^t \) be a complete set of representatives of the \( G/C \)-orbits on \( \text{Irr}(\overline{D}) \setminus \{1_D\} \), where here we view \( \text{Irr}(\overline{D}) \subseteq \text{Irr}(D) \), and with this identification \( \text{Irr}(\overline{D}) \) are exactly the elements of \( \text{Irr}(D) \) with order dividing \( p \). Note that \( \ker(\lambda_i) = \Phi(D) \) for all \( 1 \leq i \leq t \), hence \( G_{\lambda_i} = C \) for every \( 1 \leq i \leq t \), and all
the orbits of the action of $G/C$ on $\text{Irr}(D) \setminus \{1_D\}$ have the same size $m$. In particular, $t = \frac{p-1}{m}$. Since $\theta$ is $G$-invariant, for every $1 \leq i \leq t$ we have that $G_{\theta_{\lambda_i}} = G_{\lambda_i} = C$, and by the Clifford correspondence, $|\text{Irr}(G(\theta_{\lambda_i}))| = |\text{Irr}(C(\theta_{\lambda_i}))| = 1$. Also, since $G/C$ is cyclic, $\theta$ extends to $G$ and therefore by Gallagher theory $|\text{Irr}(G(\theta))| = m$. Then

$$|\text{Irr}_0(B)^{\sigma_1}| = |\text{Irr}(G(\theta))| + \sum_{i=1}^{t} |\text{Irr}(G(\theta_{\lambda_i}))| = m + t = m + \frac{p-1}{m} \leq p.$$ 

Note that if $p = 2, 3$ then $m + \frac{p-1}{m} = p$, whenever $m$ divides $p - 1$. Also notice that $|\text{Irr}_0(B)^{\sigma_1}| = 1$ if, and only if, $D = 1$. \hfill $\square$

Theorem 1.4 suggests that the natural version of Theorem A for larger primes could be $|\text{Irr}_p(B_0)^{\sigma_1}| \leq p$ if, and only if, $P$ is cyclic. However, one can also find counterexamples to this statement. For instance, for $p = 11$, the semidirect product $H = \mathbb{F}_2^2 \rtimes \text{SL}_2(5)$ satisfies $|\text{Irr}_{11}(B_0(H))^{\sigma_1}| = |\text{Irr}(H)| = 10$. (We would like to thank Gabriel Navarro for providing us with this counterexample.)

We will need the following divisibility result, which we obtain by adapting the proof of [Gow79, Theorem 5.2].

**Lemma 1.5.** Let $G$ be a finite group, let $p \in \{2, 3\}$, and let $B$ be a block of $G$ with nontrivial defect group $D$. Then $p$ divides $|\text{Irr}_0(B)^{\sigma_1}|$.

**Proof.** Write

$$\psi = \sum_{\chi \in \text{Irr}(B)} \chi(1) \chi$$

and notice that $\psi$ is a character of $G$ that vanishes on $p$-singular elements by the weak block orthogonality relation (see [Nav98, Corollary 3.7]). In particular, if $P \in \text{Syl}_p(G)$, then $\psi_P(x) = 0$ whenever $1 \neq x \in P$, and consequently $\psi_P = f \rho_P$ for some natural number $f$, where $\rho_P$ denotes the regular character of $P$.

Let $\text{Irr}(B) = \{\chi_1, \ldots, \chi_t\}$ and write $\chi_i(1) = p^{a_i + h_i} b_i$, where $h_i \geq 0$ is the height of $\chi_i$ and $p$ does not divide $b_i$, for $1 \leq i \leq t$. Arrange the elements in $\text{Irr}(B)$ in such a way that $\text{Irr}_0(B) = \{\chi_1, \ldots, \chi_k\}$. By [Nav98, Theorem 3.28] we have that $\psi(1) = p^{2a-d} c$, where $|P| = p^a$, $|D| = p^d$ and $c$ is a non-negative integer relatively prime to $p$. Thus

$$p^{2a-d} c = \psi(1) = p^{2a-2d} \sum_{i=1}^{k} b_i^2 + p^{2a-2d} \sum_{j=k+1}^{t} p^{2h_j} b_j^2,$$

where $h_j \geq 1$ for all $k + 1 \leq j \leq t$. Hence

$$p^d c = \sum_{i=1}^{k} b_i^2 + \sum_{j=k+1}^{t} p^{2h_j} b_j^2,$$
and as \( d \geq 1 \), we obtain \( \sum_{i=1}^{k} b_i^2 \equiv 0 \mod p \). Since \( p \in \{2, 3\} \), we have that \( b_i^2 \equiv 1 \mod p \) for every \( 1 \leq i \leq k \), and hence \( k \) is divisible by \( p \).

Recall that the group \( A = \langle \sigma_1 \rangle \) acts on \( \text{Irr}_0(B) \), and as such, we may view \( A \) as having order a power of \( p \). Since \( |\text{Irr}_0(B)^{\sigma_1}| = |\text{Irr}_0(B)^A| \), we obtain that \( p \) divides \( |\text{Irr}_0(B)^{\sigma_1}| \) by the class equation for group actions. \( \square \)

The conclusion of the result above does not hold if \( p > 3 \), as the dihedral group \( D_{2p} \) provides a counterexample. Indeed, \( D_{2p} \) has a unique \( p \)-block and every irreducible character has \( p' \)-degree and is \( \sigma_1 \)-fixed. Hence \( |\text{Irr}_{p'}(B_0(D_{2p}))^{\sigma_1}| = |\text{Irr}(D_{2p})| = 2 + (p-1)/2 < p \).

Finally, we prove the main result of this section.

**Theorem 1.6.** Let \( p \in \{2, 3\} \). Let \( G \) be a finite group and let \( B \) be a \( p \)-block of \( G \) with a nontrivial normal defect group \( D \). Then

\[ |\text{Irr}_0(B)^{\sigma_1}| = p \text{ if, and only if, } D \text{ is cyclic.} \]

In particular, Conjecture B follows from the Alperin–Galois–McKay conjecture.

**Proof.** By Theorem 1.4 we know that the “if” implication holds. We now assume that \( |\text{Irr}_0(B)^{\sigma_1}| = p \) and we work to show that \( D \) is cyclic.

Write \( C = C_G(D) \) and let \( \theta \in \text{Irr}(CD) \) be the canonical character of \( B \). Let \( \{\lambda_i\}_{i=1}^t \) be a complete set of representatives of the \( G/CD \)-orbits on \( \text{Irr}(D/\Phi(D)) \setminus \{1_D\} \). By Theorem 1.3 we may assume that \( G_\theta = G \) and

\[ \text{Irr}_0(B)^{\sigma_1} = \bigcup_{i=1}^t \text{Irr}(G|\theta_{\lambda_i}) \]

is a disjoint union. If \( p = 2 \), then

\[ 2 = |\text{Irr}_0(B)^{\sigma_1}| = |\text{Irr}(G|\theta)| + \sum_{i=1}^t |\text{Irr}(G|\theta_{\lambda_i})| . \]

Since \( D \) is nontrivial by hypothesis, we have that \( t \geq 1 \). Thus \( t = 1 \) and the characters \( \theta \) and \( \theta_{\lambda_1} \) are fully ramified with respect to their inertia subgroups. In particular, there are positive integers \( e \) and \( e_1 \) such that \( |G : C| = e^2 \) and \( |G_{\theta_{\lambda_1}} : C| = e_1^2 \). Suppose that \( |D| = 2^n \). Since \( G/CD \) acts transitively on the nontrivial elements of \( D/\Phi(D) \), we have that

\[ 2^n - 1 = |G : G_{\lambda_1}| = \left( \frac{e}{e_1} \right)^2 = f^2 . \]

The equality \( f^2 + 1 = 2^n \) forces \( f \) to be odd, then \( f^2 \equiv 1 \mod 8 \), and so \( f^2 + 1 \equiv 2 \mod 8 \) leaves as the only possibility \( n = 1 = f \), that is, \( D = C_2 \), as wanted. These techniques do not totally suffice to prove the case where \( p = 3 \). We first need to show that we may assume \( \Phi(D) = 1 \). Indeed, write \( \overline{G} = G/\Phi(D) \), \( \overline{D} = D/\Phi(D) \).
and let \( \mathcal{B} \) be a block of \( \overline{G} \) contained in \( B \) such that \( \overline{D} \) is the defect group of \( \mathcal{B} \) by [Nav98, Theorem 9.9]. Then \( \text{Irr}_0(\mathcal{B})^{\sigma_1} \subseteq \text{Irr}_0(B)^{\sigma_1} \). By Theorem 1.3 we have that \( \text{Irr}_0(\mathcal{B})^{\sigma_1} = \text{Irr}(\mathcal{B}) \) is non-empty. Hence by Theorem 1.5, we have that \( p \) divides \( |\text{Irr}_0(\mathcal{B})^{\sigma_1}| = |\text{Irr}_0(B)| = p \). If \( \Phi(D) \neq 1 \) we can apply induction to obtain that \( \overline{D} \) is cyclic, and thus \( D \) is cyclic. Hence we may assume that \( \Phi(D) = 1 \). Since \( D \) is \( p \)-elementary abelian, then \( \text{Irr}_0(B)^{\sigma_1} = \text{Irr}(B) \) by the description of these sets in Equation (1) and Theorem 1.3. By [Sam14, Proposition 15.2], if \( p = 3 \) then \( |\text{Irr}(B)| = p \) implies \( |D| = p \) and the proof is finished. \( \square \)

2. Reducing to simple groups

The aim of this section is to reduce the statement of Theorem A to a problem on simple groups that we will solve in Section 3.

2.1. Preliminaries. We start these preliminaries by recalling some classical facts about groups with a cyclic Sylow \( p \)-subgroup. The following is known as Burnside’s \( p \)-complement theorem. It appeared in [Bur95, Note III], although Burnside himself acknowledged that the result was already contained in the work of Frobenius in [Fro93].

**Theorem 2.1.** Suppose that a finite group \( G \) has cyclic Sylow \( p \)-subgroups, where \( p \) is the smallest prime divisor of its order. Then \( G \) has a normal \( p \)-complement.

In particular, finite groups having cyclic Sylow 2-subgroups possess a normal 2-complement. More generally, by work of [Her70] and [Bra76] (see [Bra76, Proposition 2C] and [Bra76, Theorem 3C]), if \( G \) has a cyclic Sylow \( p \)-subgroup, then either \( G \) is \( p \)-solvable or it has the following structure.

**Theorem 2.2** (Brauer). If \( G \) is a finite group that is not \( p \)-solvable and has a cyclic Sylow \( p \)-subgroup, then any normal subgroup \( W \triangleleft G \) has either order not divisible by \( p \) or index in \( G \) not divisible by \( p \).

We continue the preliminaries with results concerning the action of Galois automorphisms on characters belonging to principal blocks. Recall that \( \chi \in \text{Irr}(B_0(G)) \) if, and only if,

\[
\sum_{x \in G^0} \chi(x) \neq 0,
\]

where \( G^0 \) is the subset of elements of \( G \) of order not divisible by \( p \). Some properties of characters in the principal block are listed below.

**Lemma 2.3.** Let \( G \) be a finite group, and let \( N \triangleleft G \).

(a) We have that \( \text{Irr}(B_0(G/N)) \subseteq \text{Irr}(B_0(G)) \), with equality whenever \( N \) is a \( p' \)-group.

(b) If \( H_i \) are finite groups and \( \gamma_i \in \text{Irr}(B_0(H_i)) \), for \( i = 1, \ldots, t \), then \( \gamma_1 \times \cdots \times \gamma_t \in \text{Irr}(B_0(H_1 \times \cdots \times H_t)) \).
(c) Suppose that $\theta \in \text{Irr}_p(B_0(N))$ extends to $NP$, where $P \in \text{Syl}_p(G)$. Then there exists $\chi \in \text{Irr}_p(B_0(G))$ over $\theta$.

Proof. This is [NT19, Lemma 2.2].

We summarize below some results obtained in Section 1, here stated with respect to the principal block. The first part was first observed by G. Navarro (in private communication).

**Theorem 2.4.** Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$.

(a) If $P$ is normal in $G$, then $|\text{Irr}_p(B_0(G))^{\sigma_1}| = |\text{Irr}(G/O_{p'}(G)\Phi(P))|$. 
(b) If $P$ is cyclic, then $1 \leq |\text{Irr}_p(B_0(G))^{\sigma_1}| \leq p$.
(c) If $P$ is nontrivial and $p \in \{2,3\}$, then $|\text{Irr}_p(B_0(G))^{\sigma_1}| \neq 0$ is divisible by $p$.

Proof. To prove part (a), assume that $P \trianglelefteq G$. Then $G$ is $p$-solvable and by Fong’s theorem [Nav98, Theorem 10.20] $\text{Irr}(B_0(G)) = \text{Irr}(G/O_{p'}(G))$. Hence we may assume that $O_{p'}(G) = 1$ and, in particular, $C_G(P) \leq P$. By Theorem 1.3

$$\text{Irr}_p(B_0(G))^{\sigma_1} = \text{Irr}_p(G)^{\sigma_1} = \bigcup_{\lambda \in \text{Irr}(P/\Phi(P))} \text{Irr}(G|\lambda) = \text{Irr}(G/\Phi(P)).$$

Part (b) is a straightforward application of Theorem 1.4. Part (c) is a direct consequence of Lemma 1.5.

The following is an application of the relative Glauberman correspondence, as stated in [NTV19, Theorem E], that will be useful in our context.

**Lemma 2.5.** Let $N \triangleleft M$ and let $p$ be a prime. Assume that $M/N$ is a $p'$-group, and a $p$-group $P$ acts on $M$ stabilizing $N$. If $\theta \in \text{Irr}_p(B_0(N))$ is $P$-invariant, then there exists some $P$-invariant $\psi \in \text{Irr}_p(M|\theta)$ lying in the principal $p$-block of $M$.

Proof. Let $C/N = C_{M/N}(P)$. Let $\psi \in \text{Irr}(B_0(C))$ lie over $\theta$ (such $\psi$ exists by [Nav98, Theorem 9.4]). Then $\psi$ is $P$-invariant, by [NTV19, Lemma 2.1]. Hence $\psi \in \text{Irr}_p(B_0(C)|\theta)^P$. (Here for a set $\Omega$ of characters acted on by $P$, we write $\Omega^P$ for the $P$-fixed points of $\Omega$.)

From the proof of the relative Glauberman correspondence [NTV19, Theorem 2.3], we obtain the following equality

$$|\text{Irr}_p(B_0(M)|\theta)^P| = |\text{Irr}_p(B_0(C)|\theta)^P|,$$

and the result follows.

Since we will need to construct $\sigma_1$-invariant characters of $p'$-degree over a given character, we will frequently use [NT19, Lemma 2.1]. We state it here for the reader’s convenience. Recall that $A = \langle \sigma_1 \rangle$ acts on the irreducible characters of a group $G$ and of any of its subgroups as a $p$-group.
Lemma 2.6. Let $H \leq G$ be finite groups, and let $A$ be a $p$-group for some prime $p$. Suppose that $A$ acts on the characters of $G$ and $H$ such that $[\chi^a, \psi^a] = [\chi, \psi]$ for all of the characters $\chi, \psi$ of $G$ (and of $H$), and such that $(\chi_H)^a = (\chi^a)_H$ for every character $\chi$ of $G$ and every $a \in A$.

(i) Let $\chi$ be an $A$-invariant character of $G$. If there exists an $A$-invariant $\psi \in \text{Irr}(H)$ with $[\chi_H, \psi]$ not divisible by $p$, then there exists some $A$-invariant $\xi \in \text{Irr}(G)$ such that $[\chi, \xi][\xi_H, \psi]$ is not divisible by $p$.

(ii) If $\chi$ is an $A$-invariant $p'$-degree character of $G$, then $\chi$ has an $A$-invariant $p'$-degree irreducible constituent $\xi$ appearing in $\chi$ with $p'$-multiplicity.

Proof. This is [NT19, Lemma 2.1].

The following result will be useful in the reduction processes that will be carried out later in this section.

Theorem 2.7. Let $N \triangleleft G$. Assume that $G/N$ has a normal $p$-complement $M/N$. Let $P \in \text{Syl}_p(G)$. If $\theta \in \text{Irr}_p(B_0(N))^{\sigma_1}$ has some $\sigma_1$-invariant extension to $PN$, then there exists some $\xi \in \text{Irr}_p((B_0(G))^{\sigma_1}$ lying over $\theta$.

Proof. By Lemma 2.5, there exists some $P$-invariant $\psi \in \text{Irr}_p(M|\theta)$ lying in the principal $p$-block of $M$. By Lemma 1.1, $\psi$ is also $\sigma_1$-invariant. Denote by $\tau \in \text{Irr}(PN)$ a $\sigma_1$-invariant extension of $\theta$. Note that $[(\tau^G)_M, \psi] = [(\theta^M, \psi) = [\psi_N, \theta]$ is not divisible by $p$. Since $\chi = \tau^G$ is $\sigma_1$-invariant and $p$ does not divide $[\chi_M, \psi]$, by Lemma 2.6(i) $\chi$ has some $\sigma_1$-invariant constituent $\xi$ with $[\xi_M, \psi]$ not divisible by $p$. In particular, $\xi$ extends $\psi$. Since $G/M$ is a $p$-group, $B_0(G)$ is the only block covering $B_0(M)$ and so $\xi \in \text{Irr}_p((B_0(G))^{\sigma_1}$ and lies over $\theta$.

We end the preliminaries with a technical result. Although it is a classical argument in the context of reduction theorems, we record it here in the exact form in which it will be applied later on.

Lemma 2.8. Let $G$ be a finite group and let $N \triangleleft G$ be a direct product of copies of a simple nonabelian group $S$ transitively permuted by $G$. Let $P \in \text{Syl}_p(G)$. If some $1_S \neq \phi \in \text{Irr}_p(B_0(S))^{\sigma_1}$ is $X$-invariant, where $X \in \text{Syl}_p(Aut(S))$, then there exists some $P$-invariant $1_N \neq \theta \in \text{Irr}_p(B_0(N))^{\sigma_1}$. In particular, if $N$ is a minimal normal subgroup of $G$, then $\theta$ extends to a $\sigma_1$-invariant irreducible character of $PN$.

Proof. If $Q$ is a Sylow $p$-subgroup of $G$, then $\mathcal{N}_Q(S)C_G(S)/C_G(S)$ is a Sylow $p$-subgroup of $\mathcal{N}_G(S)/C_G(S) \leq Aut(S)$ and, in particular, is a $p$-subgroup of Aut($S$). Choose $Q \in \text{Syl}_p(G)$ such that $\mathcal{N}_Q(S)C_G(S)/C_G(S) \leq X$. Hence $\phi$ is $\mathcal{N}_Q(S)$-invariant.

Write $N = S_1 \times S_2 \times \cdots \times S_t$, with $S_i \cong S$, in a way such that $\{S_1, \ldots, S_r\}$ is a $Q$-orbit. Without loss of generality we may assume that $S_1 = S$. Let $x_i \in Q$ be such that $S_i = S_1^{x_i}$ and consider the character $\psi_1 = \phi^{x_1} \times \phi^{x_2} \times \cdots \times \phi^{x_r} \in \text{Irr}_p(B_0(S_1 \times S_2 \times \cdots \times S_r))^{\sigma_1}$ by Lemma 2.3. Since $\phi$ is $\mathcal{N}_Q(S)$-invariant, we have
Let $N$ itself. Then $1$ allows for a much simpler reduction for the case where $p$ the fact that groups with cyclic Sylow $2$-subgroups possess a normal $2$-complement, problem on simple groups. We do this in Theorem 2.10 below. We take advantage of The case where $2.2$. Now $P$ is the determinantal order relatively prime to $p$ is the determinantal order of $q$ is the unique extension of $\hat{\psi} \in \text{Irr}_p(B_0(N))^{\sigma_1}$ is $P$-invariant.

Since $PN/N$ is a $p$-group and $N$ is perfect, we see $(\theta(1) o(\theta), [PN : N]) = 1$, where $o(\theta)$ is the determinantal order of $\theta$. By [Isa06, Corollary 6.28], $\theta$ has a canonical extension $\hat{\theta} \in \text{Irr}_p(PN)$, which is $\sigma_1$-invariant, since $\hat{\theta}$ is the unique extension of $\theta$ of determinantal order relatively prime to $p$. 

\[ \square \]

2.2. The case where $p = 2$. Here we reduce the $p = 2$ case of Theorem A to a problem on simple groups. We do this in Theorem 2.10 below. We take advantage of the fact that groups with cyclic Sylow $2$-subgroups possess a normal $2$-complement, which allows for a much simpler reduction for the case where $p = 2$ than for the case where $p = 3$. Below we collect the properties of simple groups that will be key for proving Theorem 2.10.

**Theorem 2.9.** Let $S$ be a nonabelian simple group.

(a) There exist $1_S \neq \phi_1, \phi_2 \in \text{Irr}_2(B_0(S))^{\sigma_1}$ which are not $\text{Aut}(S)$-conjugate.

(b) Let $A = \text{Aut}(S)$. Let $X \leq \text{Aut}(S)$ be such that $X/S$ is a Sylow $2$-subgroup of $A/S$. Then there exists an $X$-invariant $1_S \neq \phi \in \text{Irr}_2(B_0(S))^{\sigma_1}$.

We will prove Theorem 2.9 in Section 3.2.

**Theorem 2.10.** Let $G$ be a finite group of even order. Let $P \in \text{Syl}_2(G)$. Then

$$|\text{Irr}_2(B_0(G))^{\sigma_1}| = 2 \quad \text{if, and only if,} \quad P \text{ is cyclic.}$$

**Proof.** If $P$ is cyclic, then $G$ has a normal $2$-complement $M$ by Theorem 2.1. Then $\text{Irr}(B_0(G)) = \text{Irr}(G/M) = \text{Irr}(P)$ and the result follows by Theorem 2.4.

For the converse, we proceed by induction on $|G|$. Let $N$ be a minimal normal subgroup of $G$. By Theorem 2.9(a), $N$ is a proper nontrivial subgroup of $G$.

If $|G/N|$ is odd, then $P \subseteq N$. If $N$ is a $2$-group, then $P = N \triangleleft G$. In this case, $\text{Irr}_2'(B_0(G))^{\sigma_1} = \text{Irr}(G/\Phi(P)O_2'(G))$ by Theorem 2.4. Then the hypothesis forces $\overline{G} = G/\Phi(P)O_2'(G)$ to be a group with $2$ conjugacy classes. Hence $\overline{G} \cong C_2$, implying $|P/\Phi(P)| = 2$. In other words, $P$ is cyclic, as wanted. Otherwise, $N$ is a direct product of $n$ simple nonabelian groups isomorphic to a simple group $S$. By Theorem 2.9(a), there exist nontrivial $\phi_1, \phi_2 \in \text{Irr}_2'(B_0(S))^{\sigma_1}$ that are not $\text{Aut}(S)$-conjugate. Let $\theta_i$ be the character of $N$ corresponding to the $n$th direct product of $\phi_i$ with itself. Then $1_N \neq \theta_1, \theta_2 \in \text{Irr}_2'(B_0(N))^{\sigma_1}$ are not $G$-conjugate. In particular, if we take $\chi_i \in \text{Irr}(B_0(G)|\theta_i)$, then $\chi_1 \neq \chi_2$ and, by Lemma 1.1, both are $\sigma_1$-invariant, a contradiction.
Hence we may assume that $|G/N|$ is even. Since $\text{Irr}(B_0(G/N)) \subseteq \text{Irr}(B_0(G))$, and $|\text{Irr}_2(B_0(G/N))^{\sigma_1}| \geq 2$ by Lemma 1.5, it follows that $PN/N$ is cyclic by induction. If $N$ is a 2-group, then $P \cong PN/N$ and we are done. Suppose that $N$ is a 2-group. Since $P/N$ is cyclic, we see $|\text{Irr}_2(P/N)^{\sigma_1}| = 2$ by Theorem 2.4. If $|\text{Irr}_2(P)^{\sigma_1}| = |\text{Irr}_2(P/N)^{\sigma_1}| = 2$, then again by Theorem 2.4, we have that $|P/\Phi(P)| = 2$ and hence $P$ is cyclic. Thus, we may assume that there exists $\tau \in \text{Irr}(P)$, linear and $\sigma_1$-invariant, not containing $N$ in its kernel. Then $1_N \neq \tau_N = \theta \in \text{Irr}_2(B_0(N))^{\sigma_1}$ is $\tau$-invariant. Since $P/N$ is cyclic, $G/N$ has a normal 2-complement $M/N$ with $G/M$ cyclic and hence, by Theorem 2.7, there exists some $\chi \in \text{Irr}_2(B_0(G))^{\sigma_1}$ lying over $\theta \neq 1_N$. This would contradict the hypothesis $|\text{Irr}_2(B_0(G))^{\sigma_1}| = 2$, as $G/N$ already has two such characters.

We may therefore assume that $N$ is a direct product of $t$ copies of a simple non-abelian group $S$. By Theorem 2.9(b) and Lemma 2.8, we can find some $P$-invariant $1_N \neq \theta \in \text{Irr}_2(B_0(N))^{\sigma_1}$ that extends canonically to $PN$. Such a canonical extension is therefore $\sigma_1$-invariant too. By Theorem 2.7, there exists some $\chi \in \text{Irr}_2(B_0(G))^{\sigma_1}$ lying over $\theta \neq 1_N$, a contradiction. \hfill \square

2.3. The case where $p = 3$. Here we reduce Theorem A in the case $p = 3$ to a question on simple groups, which is done in Theorem 2.13 below. As in the $p = 2$ case, we begin by summarizing some results on simple groups that will be key for the reduction. The following will be proved in Section 3.7 below.

**Theorem 2.11.** Let $S$ be a nonabelian simple group of order divisible by 3, and let $P \in \text{Syl}_3(S)$ and $X \in \text{Syl}_3(\text{Aut}(S))$. Write $B_0 = B_0(S)$. Then:

(a) If $P$ is cyclic, then $\text{Irr}_3(B_0)^{\sigma_1} = \{1_S, \phi_1, \phi_2\}$, where the $\phi_i$ are nontrivial and not $\text{Aut}(S)$-conjugate, and some $\phi_i$ is $X$-invariant.

(b) If $P$ is not cyclic, then $\text{Irr}_3(B_0)^{\sigma_1} \supseteq \{1_S, \phi_1, \phi_2, \phi_3\}$, where the nontrivial $\phi_i$ are pairwise not $\text{Aut}(S)$-conjugate, and some $\phi_i$ is $X$-invariant.

The following result will be key for handling the case where a minimal normal subgroup of $G$ is a $p$-elementary abelian group in the reduction of Theorem A for $p = 3$ below. It holds without restrictions on the prime $p$ and it can also be seen as a consequence of [Nav98, Problem 4.8].

Given $\chi \in \text{Irr}(G)$, $p$ a prime, and $P \in \text{Syl}_p(G)$, define $\tilde{\chi}$ by $\tilde{\chi}(x) = |P|\chi(x)$ if $x \in G$ is $p$-regular and $\tilde{\chi}(x) = 0$ otherwise. Then $\tilde{\chi}$ is a generalized character of $G$ by [Nav98, Lemma 2.15].

**Lemma 2.12.** Let $G$ be a group of order divisible by a prime $p$. Let $P \in \text{Syl}_p(G)$. If $C_G(P) \subseteq P$, then whenever $\lambda$ is a linear character of $P$, every irreducible constituent of $\lambda^G$ of $p'$-degree lies in the principal $p$-block $B_0(G)$ of $G$.

**Proof.** Since $C_G(P) \subseteq P$, we have that $B_0(P)^G$ is defined by [Nav98, Theorem 4.14]. By Brauer’s third main theorem $B_0(P)^G = B_0(G)$. Let $\chi$ be an irreducible constituent
of $\chi^G$ of $p'$-degree. Suppose that $\chi \notin B_0(G)$. Note that $\chi = \bar{\chi} = \bar{\chi}(1)\rho_P$, where $\rho_P$ denotes the regular character of $P$. By [Nav98, Lemma 6.5(a)], we have

$$\chi(1) = \frac{[\bar{\chi}, \lambda]}{\lambda(1)} = 0 \mod P,$$

where $P$ is the ideal with respect to which Brauer characters (and blocks) are defined in [Nav98, Chapter 2]. In particular, $P \cap \mathbb{Z} = p\mathbb{Z}$. Hence if $\chi \notin B_0(G)$ we get $\chi(1) \equiv 0 \mod p$, a contradiction.

We can now reduce Theorem A for $p = 3$.

**Theorem 2.13.** Let $G$ be a finite group of order divisible by 3. Let $P \in \text{Syl}_3(G)$. Then

$$|\text{Irr}_{p'}(B_0(G))^{\sigma_1}| = 3 \text{ if, and only if, } P \text{ is cyclic.}$$

**Proof.** Write $p = 3$. If $P$ is cyclic, then $|\text{Irr}_{p'}(B_0(G))^{\sigma_1}| = p$ by Theorem 1.4.

Assume now that $|\text{Irr}_{p'}(B_0(G))^{\sigma_1}| = p$. We prove that $P$ is cyclic by induction on the order of $G$. First, notice that we may assume that $G$ is not simple, by Theorem 2.11, and $N_G(P) < G$ by Theorem 1.6.

**Step 1. We may assume $O_{p'}(G) = 1$.**

This follows by induction since $\text{Irr}(B_0(G/O_{p'}(G))) = \text{Irr}(B_0(G))$ by Theorem 2.3(a).

**Step 2 We may assume that $O_{p'}(G) = G$.**

Otherwise, let $M \triangleleft G$ with $|G : M|$ not divisible by $p$ and $G/M$ simple. Then $P \subseteq M$ and by the Frattini argument $MN_G(P) = G$. Hence $MC_G(P) \triangleleft G$ and therefore $G = C_G(P)M$ or $C_G(P) \subseteq M$. Suppose $G = MC_G(P)$, then restriction defines a bijection $\text{Irr}(B_0(G)) \rightarrow \text{Irr}(B_0(M))$ by [Alp76, Lemma 1.1] and [Dad77]). In particular, by Lemma 1.1 restriction also defines a bijection $\text{Irr}_{p'}(B_0(G))^{\sigma_1} \rightarrow \text{Irr}_{p'}(B_0(M))^{\sigma_1}$. In this case we are done by induction. Therefore we may assume that $C_G(P) \subseteq M$. We claim that $B_0(G)$ is the only block of $G$ covering $B_0(M)$. Indeed, let $B$ be a block of $G$ covering $B_0(M)$. By [Nav98, Theorem 9.26], we have that $P$ is a defect group of $B$. By [Nav98, Lemma 9.20], $B$ is regular with respect to $M$ and hence by [Nav98, Theorem 9.19], $B_0(M)^G = B$. By Brauer's Third Main Theorem we have that $B_0(M)^G = B_0(G)$ and hence $B = B_0(G)$ and the claim is proven. Therefore $\text{Irr}(G/M) \subseteq \text{Irr}_{p'}(B_0(G))^{\sigma_1}$. This implies that $|\text{Irr}(G/M)| = 2$ and hence $|\text{Irr}_{p'}(B_0(G))^{\sigma_1}| = 1, \lambda, \theta$ with $M \subseteq \ker(\lambda)$. Let $\tau \in \text{Irr}_{p'}(B_0(M))^{\sigma_1}$ be nontrivial by Lemma 1.5. Hence $\tau$ lies under $\theta$. Since $\theta_M$ has at most two irreducible constituents, we have that $|\text{Irr}_{p'}(B_0(M))^{\sigma_1}| = 3$ and we are done by induction.

**Step 3. If $1 \neq M \triangleleft G$, then $|\text{Irr}_{p'}(B_0(G/M))^{\sigma_1}| = p$ and every $\chi \in \text{Irr}_{p'}(B_0(G))^{\sigma_1}$ satisfies $M \subseteq \ker(\chi)$. Moreover, $PM/M$ is cyclic.**

By Step 2, we have that $p$ divides $|G : M|$ and hence the first statement follows from Lemma 1.5 and Lemma 2.3 (a). The second follows by induction.
**Step 4.** Let $N$ be a minimal normal subgroup of $G$. We may assume $PN < G$.

Suppose the contrary. By Step 1 and the fact that $N_G(P) < G$, we have that $N$ is the direct product of $n$ copies of a nonabelian simple group $S$ of order divisible by $p$. By Theorem 2.11 there exist $1_s \neq \phi \in \text{Irr}_p(B_0(S))^\sigma_1$ $X$-invariant for $X \in \text{Syl}_p(\text{Aut}(S))$. By Lemma 2.8, there is some $P$-invariant $\theta \in \text{Irr}_p(B_0(N))^\sigma_1$ and $\theta$ extends canonically to $\chi \in \text{Irr}(G)$. In particular, $\chi^{\sigma_1} = \chi$. Since $B_0(G)$ is the only block covering $B_0(N)$ by [Nav98, Corollary 9.6], we have that $\chi \in \text{Irr}_p(B_0(G))^\sigma_1$. But $N \not\subseteq \ker(\chi)$, contradicting Step 3.

**Step 5.** The group $G/N$ is perfect. In particular, $G/N$ is not $p$-solvable. Also, we may assume that $G$ itself is perfect.

Write $(G/N)' = M/N$. If $M < G$, then $p$ divides the order of $G/M$ by Step 2. By [Isa08, Theorem 5.17] then $M/N$ is a $p'$-group. Since $G/M$ is abelian, we have that $PM \triangleleft G$, so $G = PM$, again applying Step 2. Since $PN \cap M = N$, it follows that $M/N$ is a normal $p'$-complement of $G/N$.

Let $\tau \in \text{Irr}_p'(B_0(PN))^\sigma_1$. Suppose that $N \not\subseteq \ker(\tau)$. Then $\theta = \tau_N \in \text{Irr}_p'(B_0(N))^\sigma_1$ satisfies the hypothesis of Theorem 2.7. In particular, there is some $\chi \in \text{Irr}_p'(B_0(G))^\sigma_1$ lying over $\theta \neq 1_N$, again a contradiction to Step 3. Hence $\tau$ has $N$ in its kernel. Therefore, $\tau \in \text{Irr}_p'(PN/N)^\sigma_1$. Since $PN/N$ is a $p'$-group it has just one block, and hence $\tau \in \text{Irr}_p'(B_0(PN/N))^\sigma_1$. This shows that $|\text{Irr}_p'(B_0(PN))^\sigma_1| = |\text{Irr}_p'(B_0(PN/N))^\sigma_1| = p$. Since $PN < G$ by Step 4, by induction we have that $P$ is cyclic. Hence we may assume that $M = G$ and $G/N$ is perfect. In particular, $O_p(G/N) = G/N$ and by Step 2 also $O_p(G/N) = G/N$. Hence $G/N$ is not $p$-solvable.

Since $G/N$ is perfect, $G = G'N$. By minimality of $N$ we have that either $N \cap G' = N$ or $N \cap G' = 1$. If $N \cap G' = N$, then $G = G'$. If $N \cap G' = 1$, then $G$ is the direct product of $N$ and $G'$. Since $p$ divides both $|N|$ and $|G'|$ by Step 1, by Theorem 1.5 we have that $|\text{Irr}_p'(B_0(G))^\sigma_1| \geq p^2 = 9$, a contradiction.

**Step 6** We may assume that $Z(G) = 1$.

If $Z(G) \neq 1$, by Step 3 we have that $G/Z(G)$ has cyclic Sylow $p$-subgroups. By [Isa08, Corollary 5.4], $p$ does not divide $|Z(G)|$, contradicting Step 1.

**Step 7.** If $N \trianglelefteq M \triangleleft G$ and $M < G$, then $p$ does not divide the order of $M/N$.

By Step 5, $G/N$ is not $p$-solvable and, by Step 2, $p$ divides the order of $G/M$. Hence $p$ does not divide the order of $M/N$ by Step 3 and Theorem 2.2.

**Step 8.** We may assume that $N$ is semisimple of order divisible by $p$.

Otherwise $N$ is a $p$-group and thus $N \subseteq P$. We consider $N \subseteq C_G(N) \triangleleft G$. Note that $C_G(N) < G$ by Step 6. By Step 7, $C_G(N)/N$ is a $p'$-group. Then $C_G(N) = X \times N$ with $X \subseteq O_p'(G)$, forcing $X = 1$. In particular, $N$ is self-centralizing and $P = C_G(P)P$.

Since $P/N$ is cyclic by Step 3, we know by the first part of this proof that $|\text{Irr}_p'(P/N)^\sigma_1| = p$. If $\text{Irr}_p'(P)^\sigma_1 = \text{Irr}_p'(P/N)^\sigma_1$ then we have that $|\text{Irr}_p(P)^\sigma_1| = p$, that is, $|\text{Irr}(P/\Phi(P))| = p$, so that $P$ is cyclic. Therefore, we may assume that
there is a linear character $\lambda \in \text{Irr}(P)$, $\sigma_1$-invariant, such that $\lambda_N \neq 1_N$. By Lemma 2.6(ii), there exists a $p'$-degree irreducible constituent $\chi$ of $\lambda^G$ which is $\sigma_1$-invariant. By Lemma 2.12, such $\chi$ lies in $B_0(G)$. Hence we find $\chi \in \text{Irr}_{p'}(B_0(G))^{\sigma_1}$ such that $N \nsubseteq \ker(\chi)$, contradicting Step 3.

**Final Step.** By Step 8, the group $N$ is a direct product of $t$ copies of a nonabelian simple group $S$ of order divisible by $p$ that are transitively permuted by $G$. By Steps 3 and 5, we know that $G/N$ is a group with cyclic Sylow $p$-subgroups and $G/N$ is not $p$-solvable. Write $K/N = O_{p'}(G/N)$. Since $G/N$ is not $p$-solvable, then $K < G$.

Write $N = S_1 \times \cdots \times S_t$, where each $S_i \cong S$ and $G$ permutes transitively the $S_i$. Write $M = \bigcap N_G(S_i) \lhd G$. We claim that $M < G$. If $M = G$, then $N = S$ is a nonabelian simple group. If $G = C_G(S)S$, since $S \cap C_G(S) = 1$, then $G = C_G(S) \times S$. By Step 1, we know that $p$ divides $|C_G(S)|$ and $p$ divides $|S|$. By Lemma 1.5 we conclude $|\text{Irr}_{p'}(B_0(G))^{\sigma_1}| \geq p^2 = 9$, a contradiction. Hence $C_G(S)S < G$. By Step 7, $p$ does not divide $|C_G(S)S : S|$, and hence $C_G(S)S \subseteq K$. Now, $G/C_G(S)S \leq \text{Out}(S)$, which is solvable by the Schreier conjecture. This contradicts Step 5 and proves the claim.

Now, by Step 7, $M \subseteq K$. Notice that $C_G(Q) \subseteq M$, where $Q = P \cap N$. Indeed, $C_G(Q) \subseteq C_G(Q \cap S_i)$ for all $1 \leq i \leq t$. Now, if $x \in C_G(Q \cap S_i)$, then $(Q \cap S_i)^x = Q \cap S_i \subseteq S_i^x \cap S_i$, and then $S_i^x = S_i$. Thus $C_G(Q \cap S_i) \subseteq N_G(S_i)$ and $C_G(Q) \subseteq M$. In particular $C_G(P) \subseteq C_G(Q) \subseteq M \subseteq K$ and by [Nav98, Lemma 9.20], $B_0(G)$ is regular with respect to $K$. By [Nav98, Theorem 9.29], the block $B_0(K)^G$ is defined and $B_0(K)^G = B_0(G)$. We claim that $B_0(G)$ is the only block of $G$ covering $B_0(K)$. Indeed, let $B$ be a block of $G$ covering $B_0(K)$. Notice that $Q \in \text{Syl}_p(K)$ and hence $Q$ is a defect group of $B_0(K)$. By [Nav98, Theorem 9.26], there is a defect group $E$ of $B$ such that $Q = E \cap K$. In particular, $C_G(E) \subseteq C_G(Q) \subseteq K$ and by [Nav98, Theorem 9.19 and Lemma 9.20], we obtain that $B = B_0(K)^G = B_0(G)$.

By Theorem 2.11, some nontrivial $\phi \in \text{Irr}_{p'}(B_0(S))^{\sigma_1}$ is $X$-invariant, where $X \in \text{Syl}_p(\text{Aut}(S))$. By Lemma 2.8, some $1_N \neq \theta \in \text{Irr}_{p'}(B_0(N))^{\sigma_1}$ is $P$-invariant and has a canonical extension $\hat{\theta}$ to $PN$, which is $\sigma_1$-invariant too. By Theorem 2.7, there is some $\psi \in \text{Irr}_{p'}(B_0(PK))^{\sigma_1}$ over $\theta$. Let $\chi = \psi^G$. Then $\chi$ is $\sigma_1$-invariant and has $p'$-degree. By Lemma 2.6(ii), $\chi$ has some constituent $\xi \in \text{Irr}_{p'}(G)^{\sigma_1}$. Now $[\xi_{PK}, \psi] = [\xi, \psi^G] \neq 0$. In particular, $\xi$ lies over $\psi_K \in \text{Irr}(B_0(K))$. Since $B_0(G)$ is the only block covering $B_0(K)$ we conclude that $\xi \in \text{Irr}_{p'}(B_0(G))^{\sigma_1}$ lies over $\theta \neq 1_N$, contradicting Step 3.

3. Simple groups

In this Section we prove Theorem 2.9 and 2.11, thus completing the proof of Theorem A.
3.1. Some Generalities on Groups of Lie Type. Since the groups of Lie type play a large role in what follows, we begin by recalling some essentials about their blocks and characters.

Let $q$ be a power of a prime. When $G = G^F$ is the group of fixed points of a connected reductive algebraic group $G$ defined over $\mathbb{F}_q$ under a Steinberg map $F$, the set of so-called rational Lusztig series corresponding to $G^*$-conjugacy classes of semisimple elements $s \in G^*$ (i.e. elements of order relatively prime to $q$). Here $G^* = (G^*)^{F^*}$, where $(G^*, F^*)$ is dual to $(G, F)$.

With this notation, we record the following lemma, proved in [SFT18a, Lemma 3.4], which describes the action of $\mathcal{H}$ on the set of rational Lusztig series and will be useful throughout this section.

**Lemma 3.1.** Let $p$ be a prime and let $s \in G^*$ be a semisimple element. Let $f$ and $b$ be integers and let $\sigma \in \mathcal{H}$ be such that $\sigma(\xi) = \xi^{p^f}$ for all $p'$-roots of unity $\xi$ and $\sigma(\zeta) = \zeta^b$ for all $p$-power roots of unity $\zeta$. If $\chi \in \mathcal{E}(G, s)$, then $\chi^\sigma \in \mathcal{E}(G, s^{p^f s^b})$.

The characters in the series $\mathcal{E}(G, 1)$ are called **unipotent** characters, and there is a bijection $\mathcal{E}(G, s) \to \mathcal{E}(C_{G^*}(s), 1)$. Hence, characters of $\text{Irr}(G)$ may be indexed by pairs $(s, \psi)$, where $s \in G^*$ is a semisimple element, up to $G^*$-conjugacy, and $\psi \in \text{Irr}(C_{G^*}(s))$ is a unipotent character of $C_{G^*}(s)$. We remark that $C_{G^*}(s)$ may fail to be connected, in which case unipotent characters of $C_{G^*}(s)$ are taken to be those lying over a unipotent character of $(C_{G^*}(s)^{F})^F$. In particular, we will call the characters indexed by $(s, 1_{C_{G^*}(s)})$ **semisimple**, and they have degree $|G^* : C_{G^*}(s)|_q$.

Using [CE04, Theorem 9.12], it follows that when $p \nmid q$, the set $\mathcal{E}_p(G, 1) := \bigcup \mathcal{E}(G, s)$, where $s$ ranges over elements of $p$-power order in $G^*$, is a union of $p$-blocks (first shown in [BM89]) and that each such block intersects $\mathcal{E}(G, 1)$ nontrivially. Such blocks are called **unipotent blocks**.

3.1.1. A General Set-up. We will often be interested in the following situation:

Let $S$ be a simple group such that there exist $G$ a simple, simply connected algebraic group over $\mathbb{F}_q$ and $F$ a Steinberg morphism satisfying $S = G/\text{Z}(G)$ with $G := G^F$ perfect. Let $(G^*, F^*)$ be dual to $(G, F)$.

If $\text{Z}(G)$ is trivial, we define $\tilde{G} := G$. Otherwise, we further let $\iota : G \hookrightarrow \tilde{G}$ be a regular embedding as in [CE04, 15.1] and let $\iota^* : \tilde{G}^* \to G^*$ be the corresponding surjection of dual groups. Write $\tilde{G} := \tilde{G}^F$, $G^* := (G^*)^{F^*}$, and $\tilde{G}^* := (\tilde{G}^*)^{F^*}$. We may then find $F$-stable maximally split tori $T$ and $\tilde{T}$ for $G$ and $\tilde{G}$, respectively, such that $T \leq \tilde{T}$. Write $T := T^F$ and $\tilde{T} := \tilde{T}^F$.

Then $\text{Z}(\tilde{G})$ is connected, $G \triangleleft \tilde{G}$, and $\text{Z}(\tilde{G}) \cap G = \text{Z}(G)$. We will write $\tilde{S} := \tilde{G}/\text{Z}(\tilde{G})$, and note that $\text{Aut}(S)$ is generated by $\tilde{S}$ and the graph-field automorphisms. Further, the (linear) characters of $\tilde{G}/G$ are in bijection with elements of $\text{Z}(\tilde{G}^*)$, and
we have $\tilde{\chi}_s \otimes \tilde{z} = \tilde{\chi}_{sz}$, where $z \in \mathbb{Z}(\tilde{G}^*)$ corresponds to $\tilde{z} \in \text{Irr}(\tilde{G}/G)$ and for semisimple $s \in \tilde{G}^*$, $\tilde{\chi}_s$ denotes the semisimple character of $\tilde{G}$ corresponding to $s$. (See [DM91, 13.30].) It will also be useful in what follows to note that if $s \in [\tilde{G}^*, \tilde{G}^*]$ is semisimple, then the semisimple character of $\tilde{G}$ corresponding to $s$ is trivial on $\mathbb{Z}(\tilde{G})$ by [NT13, Lemma 4.4].

When $q$ is a power of $p$, we note that $\text{Irr}_{\nu'}(B_0(S)) = \text{Irr}_{\nu'}(S)$. We remark that early proofs of the fact that there is a unique block of maximal defect of $S$ in this case were found by Dagger and Dipper [Dag71, Dip80, Dip83], but this can also be seen using [CE04, 6.14, 6.15, and 6.18] and the facts that $p \nmid |\mathbb{Z}(G)|$ and $S$ is a group with a strongly split BN pair as in [CE04, 2.20].

In the case of types $A_{n-1}$ and $2A_{n-1}$, we have $S$ is $PSL_n(q)$ with $\epsilon \in \{\pm 1\}$; $G = SL_n(q)$; $\tilde{G} = GL_n^\epsilon(q)$; and $S = PGL_n(q)$. Here $\epsilon = 1$ means $S = PSL_n(q)$, $\epsilon = -1$ means $S = PSU_n(q)$, and similarly for $G$ and $\tilde{G}$. We use similar notation for other twisted types. For example, $E_6^\epsilon(q)$ will denote $E_6(q)$ for $\epsilon = +$ and $2E_6(q)$ for $\epsilon = -$.

3.2. The Case $p = 2$. Here we prove Theorem 2.9. The following, found in [NST18, Lemma 3.1], will be useful in what follows.

**Lemma 3.2** (Navarro-Sambale-Tiep). Let $G$ be a finite group. If $\chi \in \text{Irr}_{\nu'}(G)$ is real-valued, then $\chi$ belongs to $B_0(G)$.

From this we see that rational-valued characters of odd degree are always contained in $\text{Irr}_{\nu'}(B_0(G))^{s_1}$. In particular, note that an odd character degree with multiplicity one must necessarily come from a character fixed by all automorphisms and Galois automorphisms, so a nontrivial character with such a degree satisfies Theorem 2.9(b).

**Lemma 3.3.** Let $S$ be a simple sporadic group, alternating group $\mathfrak{A}_n$ with $n \geq 5$, or one of the simple groups $PSL_2(4), PSL_3(2), PSL_3(4), PSU_4(2), PSU_4(3), PSL_6^+(2), 2B_2(8), B_3(2), B_3(3), D_4(2), F_4(2), 2F_4(2)' , E_6(2), 2E_6(2), G_2(2)' , G_2(3)$, or $G_2(4)$. Then Theorem 2.9 holds for $S$.

**Proof.** For $n \geq 7$, the automorphism group of $\mathfrak{A}_n$ is the symmetric group $S_n$. Recall that every irreducible character of $S_n$ is rational-valued and that an odd-degree character of $S_n$ must restrict irreducibly to $\mathfrak{A}_n$ since it has index 2. In this case, if $n = 2^{n_1} + \cdots + 2^{n_t}$ with $n_1 < n_2 < \ldots < n_t$ is the 2-adic decomposition of $n$, then [Mac71, Corollary 1.3] yields that there are $2^{n_1 + \cdots + n_t} \geq 8$ odd-degree characters of $S_n$, whose restrictions therefore yield at least 3 nontrivial members of $\text{Irr}_{\nu'}(B_0(\mathfrak{A}_n))^{s_1}$ invariant under $\text{Aut}(\mathfrak{A}_n)$.

For the remaining groups listed, the statement can be seen using [GAP] and the GAP Character Table Library. In fact, we see that for the sporadic groups other than the Janko groups, there exist at least two nontrivial odd character degrees with multiplicity 1.  

\hfill $\square$
Proposition 3.4. Let $S$ be a simple group of Lie type defined over $\mathbb{F}_q$ with $q$ a power of an odd prime $\ell$. Then Theorem 2.9 holds for $S$.

Proof. We may assume that $S$ is not isomorphic to any of the groups in Lemma 3.3, so is as in Section 3.1.1. In this case, the Steinberg character is rational-valued and $\text{Aut}(S)$-invariant, and therefore satisfies part (b) of the statement. Hence it suffices to show that there is another member of $\text{Irr}_2(B_0(S))^{\sigma_1}$. Further, we note that if $S$ is not a Suzuki or Ree group, then unipotent characters of odd degree are rational-valued (see, e.g. [SF19, Lemma 4.4]). Hence in these cases, applying Lemma 3.2, it suffices to find another unipotent character of odd degree, when possible. By observing the explicit list of unipotent character degrees in [Car85, Section 13.9], we see that there is at least one other nontrivial odd-degree unipotent character for the exceptional groups $G_2(q), 3D_4(q), F_4(q), E_6^q(q), E_7(q),$ and $E_8(q)$. Further, for $\mathfrak{G}_2(q)$, we see from the generic character table available in [CHEVIE] that the odd degree $(q^2 - q\sqrt{3} + 1)(q^2 + q\sqrt{3} + 1)$ has multiplicity one. For $PSL_{2n}(q)$ with $n \geq 2$ and $PSL_{2n}^\pm(q)$ with $n \geq 4$, one may find at least four odd-degree unipotent characters by arguing as in the proof of [NST18, Prop. 3.4] and the second paragraph of the proof of [NST18, Theorem 3.3]. Using the well-known character table for $PSL_2(q)$, we see that all four odd-degree characters are fixed by $\sigma_1$. Further, in this case, $\text{Irr}_2(S) = \text{Irr}_2(B_0(S))$. We see from part (iii) of the proof of [NST18, Theorem 3.3] that if $S = PSL_{2n}^\pm(q)$, then the Weil character $\zeta_{3, q}^{(q+1)/2}$ is a member of $\text{Irr}_2(B_0(S))$ and is real-valued.

We are left with the case $S$ is one of $P\Omega_{2n+1}(q)$ with $n \geq 2$, or $P\Omega_{2n}^\pm(q)$ with $n \geq 4$. Let $G$ be a group of Lie type of simply connected type such that $G/\mathcal{Z}(G) \cong S$. By [CE04, Theorem 21.14], the principal 2-block $B_0(G)$ is comprised of the $\mathcal{E}(G, s)$ for $s \in G^*$ having order a power of 2. On the other hand, $G^*$ has a self-normalizing Sylow 2-subgroup by [Kon05], so any $s \in G^*$ centralizing a Sylow 2-subgroup must be a 2-element. Hence $\text{Irr}_2(G) = \text{Irr}_2(B_0(G))$. Further, since $\mathcal{Z}(G)$ is a 2-group, $\text{Irr}_2(B_0(S))$ may be identified with the set of characters in $\text{Irr}_2(B_0(G)) = \text{Irr}_2(G)$ such that $\mathcal{Z}(G)$ is in the kernel (see, for example, [SFT18b, Lemma 3.5]), and since $G$ is perfect, this is just the set $\text{Irr}_2(G)$ (see for example [GSV19, Lemma 3.4]). Hence it suffices to exhibit an odd-degree irreducible character $\chi \notin \{\text{St}_G, 1_G\}$ of $G$ fixed by $\sigma_1$. Now, let $s \in G^*$ be an element of order 2 in the center of a Sylow 2-subgroup of $G^*$. Then $C_{G^*}(s)$ contains a Sylow 2-subgroup of $G^*$, and hence any semisimple character of $G$ associated to $s$ has odd degree. Now, recall that $q$ is odd and $\sigma_1$ fixes odd roots of unity. Then since any Gelfand-Graev character is induced from a linear character of the Sylow $\ell$-subgroup of $G$, we see that all Gelfand-Graev characters of $G$ are fixed by $\sigma_1$. Hence Lemma 3.1 and [SFT18a, Lemma 3.8] imply that a semisimple character associated to $s$ is fixed by $\sigma_1$, completing the proof. □

Proposition 3.5. Let $S$ be a simple group of Lie type defined in characteristic 2. Then Theorem 2.9 holds for $S$. 
Proof. Again, we may assume that $S$ is not as in Lemma 3.3. In particular, we may keep the notation as in Section 3.1.1 and we have $\text{Irr}_G(B_0(S)) = \text{Irr}_G(S)$. If $S$ is $2B_2(q)$ or $2F_4(q)$, then the generic character tables available in CHEVIE yield the result, since $|\text{Out}(S)|$ is odd and there are at least two distinct degrees of nontrivial odd-degree characters whose values are fixed by $\sigma_1$.

Otherwise, we may take the Steinberg endomorphism on $G$ to be $F = F_q \circ \tau$, where $F_q$ is the standard Frobenius induced by the map $x \mapsto x^q$ and $\tau$ is some graph automorphism. Write $\sigma := q^{|\tau|} = 2^{2m}$ with $m$ odd and let $X \leq \text{Aut}(S)$ such that $X/S \in \text{Syl}_2(\text{Out}(S))$.

Since $q$ is a power of 2, we have $Z(G) = 1$ and $\tilde{G} = G$ unless $S$ is one of $PSL^+_n(q)$ or $E^+_6(q)$. In the latter cases, $G = [\tilde{G}, \tilde{G}]$. In any case, since $\tilde{G}/G$ has odd order, we may view $X/S$ as generated by $F_{2m}$ and graph automorphisms.

Now, if $m > 1$, then the proof of [SFT18a, Lemma 6.4] (and taking into account the remark after [SFT18b, Proposition 6.5]) yields a member of $\text{Irr}_G(S)$ invariant under $X$ which is the restriction to $G$ of a semisimple character of $\tilde{G}$ trivial on $Z(G)$. Since semisimple elements have odd order and $\sigma_1$ fixes odd roots of unity, Lemma 3.1 shows that this character is also fixed by $\sigma_1$. If $m = 1$, we may similarly obtain an $X$-invariant member of $\text{Irr}_G(S)$ fixed by $\sigma_1$ by arguing as in [SFT18a, Lemma 6.4] and the remark after [SFT18b, Proposition 6.5] but using an element of $F^x_{2^m}$ of order 3 rather than an element of $F^x_q$ of order 5. This completes the proof of part (b).

It therefore suffices to verify part (a) of the statement. For $S = G_2(q), F_4(q), 2D_4(q), E_7(q)$, or $E_8(q)$, the list of character degrees at [Lüb07] yields at least two distinct nontrivial odd character degrees, completing the proof of part (a) in these cases, since by [Mal07, Theorem 6.8], odd-degree characters are semisimple (recall that we may assume $q \neq 2$ when $S = G_2(q)$ or $F_4(q)$), and hence fixed by $\sigma_1$ using Lemma 3.1.

Now, in the remaining cases, $S$ is a classical group or $E^+_6(q)$. Here $\tilde{G}^x \cong \tilde{G}$. In the case $S = PSL_2(q)$ or $PSL_3(q)$, we see that there are at least two odd-degree characters with different degrees that are fixed by $\sigma_1$, using the generic character tables available in [CHEVIE]. If $\tilde{G} = GL^+_n(q), Sp_n(q)$, or $O^+_n(q)$ with $n \geq 4$ and $n$ even in the latter two cases, let $s_1$ and $s_2$ be elements of $\tilde{G}$ with eigenvalues $\{\delta, \delta^{-1}, 1, \ldots, 1\}$ and $\{\delta, \delta, \delta^{-1}, \delta^{-1}, 1, \ldots, 1\}$, respectively, where $1 \neq \delta \in F^x_q$.

Then $s_1$ and $s_2$ are not $\text{Aut}(S)$-conjugate, and hence the corresponding semisimple characters of $\tilde{G}$ have odd degree, are not $\text{Aut}(S)$-conjugate, and are fixed by $\sigma_1$ by Lemma 3.1. Further, if $\tilde{G} = GL^+_n(q)$, semisimple classes are determined by the eigenvalues, and $Z(\tilde{G})$ is comprised of scalar matrices, so we see for $i = 1, 2$, $s_i$ is not conjugate to $s_iz$ for any $z \in Z(\tilde{G})$ unless possibly if $n = 6$. In this case, we may assume $q \neq 2$ using Lemma 3.3 and instead take $\delta \in F^x_{q^2}$ to have order at least 5, again yielding $s_i$ is not conjugate to $s_iz$ for any $z \in Z(\tilde{G})$. In any case, the corresponding
semisimple characters therefore restrict irreducibly to $G$ and are trivial on $\mathbf{Z}(\tilde{G})$ since $s_i \in [\tilde{G}, \tilde{G}] = G$. This completes part (a) in these cases.

Finally, let $S$ be $E_6^+(q)$ with $q > 2$. Then we may argue analogously to [GRS19, Proposition 4.3] to find elements $s_1$ and $s_2$ in $\tilde{G}$ with $|C_{\tilde{G}}(s_1)|_2 \neq |C_{\tilde{G}}(s_2)|_2$ such that the corresponding semisimple characters (which again must be fixed by $\sigma_1$) are irreducible on $G$ and trivial on $\mathbf{Z}(\tilde{G})$. (Indeed, we may replace the $\delta$ used there with a $\delta \in F_q^\times$ such that $3 \nmid |\delta|$.)

Theorem 2.9 now follows by combining Propositions 3.4 and 3.5 with Lemma 3.3.

3.3. The Case $p = 3$. Here we prove Theorem 2.11. We begin by stating the following classification of simple groups with cyclic Sylow 3-subgroups.

**Proposition 3.6.** Let $S$ be a finite nonabelian simple group with order divisible by 3. Then $S$ has a cyclic Sylow 3-subgroup if and only if $S$ is one of:

- The alternating group $\mathfrak{A}_5$;
- The sporadic simple group $J_1$;
- $PSL_2(q)$ for $3 \nmid q$;
- $PSL_3(q)$ for $3 \mid (q + 1)$;
- $PSU_3(q)$ for $3 \mid (q - 1)$.

**Proof.** The main result of [SZ16] yields a classification of simple groups $S$ and primes $p$ such that $S$ has an abelian Sylow $p$-subgroup. In particular, if $p = 3$, then such a simple group must be of the form $\mathfrak{A}_n$ with $n < 9$, one of a short list of sporadic simple groups, $PSL_2(q)$, $PSL_n(q)$ for $3 \mid (q + 1)$ and $n = 3, 4, 5$, $PSU_n(q)$ for $3 \mid (q - 1)$ and $n = 3, 4, 5$ or $PSp_4(q)$ with $3 \nmid q$.

Using the Atlas [CCNPW] and since $\mathfrak{A}_6$ has a noncyclic Sylow 3-subgroup and can be viewed as a subgroup of $\mathfrak{A}_n$ for $n \geq 7$, we see that the only simple alternating or sporadic groups with cyclic Sylow 3-subgroups are $\mathfrak{A}_5$ and the Janko group $J_1$. The remaining possibilities are of the form $G/\mathbf{Z}(G)$ for $G$ a classical group $SL_n(q)$ or $SU_n(q)$ with $n < 6$, or $Sp_4(q)$. Further, except in the cases of $PSL_3(q)$ and $PSU_3(q)$ listed in the statement, $|\mathbf{Z}(G)|$ is relatively prime to 3, and hence $S$ has a cyclic Sylow 3-subgroup if and only if $G$ does. Further, for the cases $G = SL_n(q)$ or $SU_n(q)$ with $n = 3, 4, 5$ under consideration, we may view the Sylow subgroup as a Sylow subgroup of $\tilde{G} = GL_n(q)$ or $GU_n(q)$, respectively, since the index of $G$ in $\tilde{G}$ is not divisible by 3.

Now, using the description of the Sylow subgroups of classical groups in [CF64, Wei55], we see that the Sylow subgroups of $GL_4(q), GU_4(q), GL_5(q), GU_5(q)$, and $Sp_4(q)$ are direct products of Sylow subgroups of at least two lower-rank groups, and hence the Sylow 3-subgroup of $G$ is not cyclic.

We are therefore left with the cases $PSL_3(q)$ with $3 \mid (q + 1)$, $PSU_3(q)$ with $3 \mid (q - 1)$, and $PSL_2(q)$. In the first two cases, and in the case $PSL_2(q)$ with $3 \nmid q$,
we may explicitly construct a cyclic Sylow 3-subgroup. Finally, if \( S = PSL_2(q) \) with \( 3 \mid q \), the Sylow 3-subgroup can be identified with the unipotent radical of \( SL_2(q) \), which is not cyclic unless \( q = 3 \), contradicting that \( S \) is simple.

Our goal in the remainder of this section is to prove the following, from which we obtain Theorem 2.11 as a corollary.

**Theorem 3.7.** Let \( S \) be a nonabelian simple group with order divisible by 3.

(i) If \( S \) has a cyclic Sylow 3-subgroup, then there exist nontrivial \( \chi_1, \chi_2 \in \text{Irr}_S(B_0(S))^{\sigma_1} \) such that \( \chi_1 \) extends to \( \text{Aut}(S) \).

(ii) If \( S \) does not have a cyclic Sylow 3-subgroup and is not a group of Lie type defined in characteristic 3, then there exist nontrivial \( \chi_1, \chi_2, \chi_3 \in \text{Irr}_S(B_0(S))^{\sigma_1} \) such that \( \chi_1 \) and \( \chi_2 \) extend to \( \text{Aut}(S) \).

- In this case, if \( S \) is further not one of \( \mathfrak{A}_6, \mathfrak{A}_7, 2F_4(2)', PSL_n(q) \) with \( n \leq 4 \), or \( PSp_4(2^{m}+1) \), then there exist nontrivial \( \chi_1, \chi_2, \chi_3 \in \text{Irr}_S(B_0(S))^{\sigma_1} \) such that \( \chi_i \) each extend to \( \text{Aut}(S) \).

(iii) If \( S \) is a group of Lie type in characteristic 3, then there exist nontrivial \( \chi_1, \chi_2, \chi_3 \in \text{Irr}_S(B_0(S))^{\sigma_1} \) that are pairwise not \( \text{Aut}(S) \)-conjugate and such that \( \chi_1 \) is invariant under \( X \), where \( X/S \in \text{Syl}_3(\text{Aut}(S)/S) \).

We first consider Theorem 3.7 for sporadic and alternating groups, as well as some “small” groups of Lie type. For two positive integers \( n \) and \( m \), we will use \( n \mid m \) to mean that \( n \mid m \) and \( \text{gcd}(n, \frac{m}{n}) = 1 \).

**Proposition 3.8.** Theorem 3.7 holds for the sporadic simple groups, \( G_2(3), 2F_4(2)', B_3(3), G_2(2)' = PSU_3(3), PSU_4(3) \), and the alternating groups \( \mathfrak{A}_n \) with \( n \geq 5 \).

**Proof.** Since the result can be seen directly using GAP for the other cases, we may assume \( S = \mathfrak{A}_n \) with \( n > 10 \). In this case, \( S \) does not have a cyclic Sylow 3-subgroup and satisfies \( \text{Aut}(S) = \mathfrak{S}_n \), where \( \mathfrak{S}_n \) denotes the corresponding symmetric group.

The characters of \( \mathfrak{S}_n \) are rational-valued and parametrized by partitions of \( n \), with their degrees given by the hook formula. Further, two characters lie in the same 3-block if and only if they have the same 3-core. We also know that \( \chi \in \text{Irr}(\mathfrak{S}_n) \) corresponding to the partition \( \lambda \) restricts irreducibly to \( \mathfrak{A}_n \) if and only if the partition is not self-conjugate. Table 1 lists the partitions and character degrees for three characters in \( \text{Irr}_S(\mathfrak{S}_n) \) that restrict irreducibly to \( \mathfrak{A}_n \), completing the proof.

### 3.3.1. Lie Type in Cross-characteristic for \( p = 3 \)

In this section, we prove Theorem 3.7 for groups of Lie type in non-defining characteristic. That is, we deal with the case \( S \) is of the form \( G/Z(G) \) for \( G \) a finite group of Lie type of simply connected type defined over a field \( \mathbb{F}_q \) with \( 3 \nmid q \). (Given Proposition 3.8, this will complete the proof of parts (i) and (ii) of Theorem 3.7.)
Table 1. Some Members of \( \text{Irr}_{3'}(B_0(\mathfrak{s}_n)) \) Irreducible on \( \mathfrak{a}_n \), \( n > 10 \)

<table>
<thead>
<tr>
<th>Condition on ( n )</th>
<th>Partition</th>
<th>( \chi(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \mid n )</td>
<td>( (1, n-1) )</td>
<td>( n-1 )</td>
</tr>
<tr>
<td>( 3 \mid n )</td>
<td>( (1, 1, n-2) )</td>
<td>( (n-1)(n-2)/2 )</td>
</tr>
<tr>
<td>( 3 \mid n ), ( 3^2 \mid (n-2) ), or ( 3 \mid (n-1) )</td>
<td>( (3, n-3) )</td>
<td>( n(n-1)(n-5)/6 )</td>
</tr>
<tr>
<td>( 3^2 \mid n ), ( 3 \mid (n-2) ), or ( 3 \mid (n-1) )</td>
<td>( (1^4, n-3) )</td>
<td>( (n-1)(n-2)(n-3)/6 )</td>
</tr>
<tr>
<td>( 3 \mid (n-1) )</td>
<td>( (2, n-2) )</td>
<td>( n(n-3)/2 )</td>
</tr>
<tr>
<td>( 3^2 \mid (n-1) )</td>
<td>( (1,2, n-3) )</td>
<td>( n(n-2)(n-4)/3 )</td>
</tr>
<tr>
<td>( 3^2 \mid (n-1) )</td>
<td>( (1^4, 2, n-5) )</td>
<td>( n(n-2)(n-3)(n-4)(n-6)/30 )</td>
</tr>
<tr>
<td>( 3 \mid (n-2) )</td>
<td>( (1^{n-4}, 2, 2) )</td>
<td>( n(n-3)/2 )</td>
</tr>
<tr>
<td>( 3 \mid (n-2) )</td>
<td>( (1^{n-2}, 2) )</td>
<td>( n-1 )</td>
</tr>
</tbody>
</table>

We will use \( \Phi_m \) to denote the \( m \)th cyclotomic polynomial in the variable \( q \). Note that using e.g. [Mal07, Lemma 5.2], \( 3 \) divides \( \Phi_m \) if and only if \( m = 3'd \) for some \( i \geq 0 \), where \( d \) is the order of \( q \) modulo 3, and in this case \( 3 \mid \Phi_m \) unless \( m = d \).

**Proposition 3.9.** Let \( S \) be a simple group of Lie type defined over \( \mathbb{F}_q \) with \( 3 \nmid q \) and assume \( S \) is not one of the groups \( \text{PSL}_n^\pm(q) \) with \( n \leq 3 \). Then there exist three nontrivial characters \( \chi_1, \chi_2, \chi_3 \in \text{Irr}_{3'}(B_0(S))^{\sigma_1} \) such that \( \chi_1 \) and \( \chi_2 \) extend to \( \text{Aut}(S) \).

Further, if \( S \) is not \( \text{PSL}_n^\pm(q) \) nor \( \text{PSp}_4(2^a) \) with \( a \) odd, then \( \chi_1, \chi_2, \chi_3 \) may be chosen to extend to \( \text{Aut}(S) \).

**Proof.** We may assume that \( S \) is not isomorphic to one of the groups considered in Proposition 3.8. Keep the notation and considerations for \( G, \tilde{G}, \tilde{T}, \) and \( \tilde{S} \) from Section 3.1. By the work of Lusztig [Lus88], the unipotent characters of \( \tilde{G} \) are trivial on \( \mathbb{Z}(\tilde{G}) \) and restrict irreducibly to \( G \). Further, when viewed as characters of \( \tilde{S} \), they are extendible to \( \text{Aut}(S) \), by [Mal08, Theorems 2.4 and 2.5], aside from some specific exceptions. The only unipotent characters which take irrational values occur for exceptional groups and have values in \( \mathbb{Q}(\sqrt{-1}, \zeta_3, \zeta_5, \sqrt{q}) \), where \( \zeta_3 \) and \( \zeta_5 \) are 3rd and 5th roots of unity, respectively, by [Gec03, Prop 5.6 and Table 1]. In any case, the unipotent characters are \( \sigma_1 \)-invariant, since \( \sqrt{q} \) is a sum of roots of unity of order relatively prime to 3.

Let \( d \) be the order of \( q \) modulo 3. In particular, we have \( d = 1 \) or 2. If \( d = 1 \), unipotent characters of degree relatively prime to 3 are constituents of the Harish-Chandra induced character \( R_{\tilde{T}}^\tilde{G}(1) \) using [Mal07, Corollary 6.6]. Further, by [Eng00, Theorem A], all members of \( R_{\tilde{T}}^\tilde{G}(1) \) lie in the same block, namely \( B_0(\tilde{G}) \).

If \( d = 2 \), then the centralizer of a Sylow \( d \)-torus is a maximal torus, using e.g. [MS16, Lemma 3.2]. Unipotent blocks of \( \tilde{G} \) are parametrized by certain \( \tilde{G} \)-conjugacy classes of pairs \( (\tilde{L}, \lambda) \) where \( \tilde{L} \) is a \( d \)-split Levi subgroup of \( \tilde{G} \) and \( \lambda \) is a \( d \)-cuspidal unipotent character of \( \tilde{L} \), by [Eng00, Theorem A]. Further, a unipotent character in
the block parametrized by \((\hat{L}, \lambda)\) can have 3'-degree only when \(\hat{L}\) is the centralizer of a Sylow 3-torus, using [Mal07, Corollary 6.6]. This yields that again in the case \(d = 2\), there is a unique block of \(\hat{G}\) containing unipotent characters of 3'-degree.

Hence when \(3 \nmid q\), every unipotent character in \(\text{Irr}_{3'}(\hat{G})\) is a member of \(\text{Irr}_{3'}(B_0(\hat{G}))^{\sigma_1}\), and restricts to a member of \(\text{Irr}_{3'}(B_0(G))^{\sigma_1}\) trivial on the center. Then this restriction may be viewed as an element of \(\text{Irr}_{3'}(B_0(S))^{\sigma_1}\), using e.g. [CE04, Lemma 17.2].

In particular, since the Steinberg character has degree a power of \(q\), it suffices to find two more unipotent characters of 3'-degree that are not one of the exceptional cases in [Mal08, Theorem 2.5]. In what follows, we will use the notation and degrees for unipotent characters as in [Car85, Sections 13.8 and 13.9].

**Exceptional Types.** In the case that \(S\) is an exceptional group of Lie type defined over \(\mathbb{F}_q\) with \(3 \nmid q\), we list in Table 2 two unipotent characters invariant under \(\text{Aut}(S)\) that have degree relatively prime to 3, completing the proof in this case.

**Types** \(A_n-1\) and \(2A_n-1\), \(n \geq 4\). In this case, let \(S\) be \(PSL_n^\epsilon(q)\) with \(n \geq 4\) and \(\epsilon \in \{\pm 1\}\). Write \(e \in \{1, 2\}\) for the number such that \(q \equiv \epsilon e\) (mod 3). That is, \(e\) is the order of \(\epsilon q\) modulo 3. Two unipotent characters are in the same 3-block of \(\hat{G} = GL_n^\epsilon(q)\) if and only if they have the same \(e\)-core (see [FS82]). For \(n \geq 5\) or for \((n, e) = (4, 2)\), the unipotent characters described in Table 3 are \(\text{Aut}(S)\)-invariant members of \(\text{Irr}_{3'}(B_0(S))^{\sigma_1}\).

Now assume \(n = 4\) and \(e = 1\). Then the unipotent character in the last line of Table 3 is an \(\text{Aut}(S)\)-invariant member of \(\text{Irr}_{3'}(B_0(S))^{\sigma_1}\). In this case, \(1_S, \text{St}_S\), and the character listed are the only unipotent characters in \(\text{Irr}_{3'}(S)\). However, since \(e = 1\), we see that every unipotent character is a member of the principal block of \(\hat{G}\), which means that \(E(\hat{G}, 1)\) is comprised of only one block. Let \(\zeta \in \mathbb{F}_q^\times\) with order 3. Then taking \(s\) to be the element \(\text{diag}(\zeta, \zeta, \zeta, 1)\) of \(\hat{G}^* \cong GL_4^\epsilon(q)\), the semisimple character \(\chi_s \in E(\hat{G}, s)\) lies in the principal block of \(\hat{G}\) and is trivial on \(Z(\hat{G})\) since \(s \in SL_4^\epsilon(q) \cong [\hat{G}^*, \hat{G}^*]\). Further, we see using Lemma 3.1 that \(\chi_s\) is fixed by \(\sigma_1\).

Since \(C_{\hat{G}^*}(s) \cong GL_2^\epsilon(q) \times GL_2^\epsilon(q)\), we see \(\chi_s(1) = (q + \epsilon)(q^2 + 1)\). Further, since the semisimple classes of \(G\) are determined by their eigenvalues and \(Z(\hat{G})\) is comprised of scalar matrices, we see that \(s\) is not conjugate to \(sz\) for any nontrivial \(z \in Z(\hat{G})\). Hence \(\chi_s|_G\) is irreducible, by the second-to-last paragraph of Section 3.1, and is therefore a member of \(\text{Irr}_{3'}(B_0(G))\), since the principal block of \(G\) is the only block covered by the principal block of \(\hat{G}\). But since \(Z(G) \leq Z(\hat{G})\) is in the kernel of \(\chi_s\), this character is therefore a member of \(\text{Irr}_{3'}(B_0(G/Z(G)))^{\sigma_1} = \text{Irr}_{3'}(B_0(S))^{\sigma_1}\), again using [CE04, Lemma 17.2]. Note that this character is not \(\text{Aut}(S)\)-conjugate to \(1_S, \text{St}_S\), nor the unipotent character labeled by \((2, 2)\), which completes the proof for \(S = PSL_n^\epsilon(q)\) with \(n \geq 4\).
Theorem 2.5). Note: 3 \nmid q

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition on d</th>
<th>Character (Notation from [Car85, 13.9])</th>
<th>(\chi(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_2(q))</td>
<td>(d = 1)</td>
<td>(\phi_{2,2}) (\phi_{1,3'})</td>
<td>(\frac{1}{7}q\Phi_2^2\Phi_6) (\frac{1}{7}q\Phi_3\Phi_6)</td>
</tr>
<tr>
<td>(G_2(q))</td>
<td>(d = 2)</td>
<td>(G_2[1]) (\phi_{1,3'})</td>
<td>(\frac{1}{7}q\Phi_1^2\Phi_6) (\frac{1}{7}q\Phi_3\Phi_6)</td>
</tr>
<tr>
<td>(3D_4(q))</td>
<td>(d = 1, 2)</td>
<td>(\phi_{1,3'}) (\phi_{1,3'})</td>
<td>(q\Phi_{12}) (q'\Phi_{12})</td>
</tr>
<tr>
<td>(F_4(q))</td>
<td>(d = 1)</td>
<td>(\phi_{4,1}) (\phi_{8,3'})</td>
<td>(\frac{7}{5}q\Phi_4^2\Phi_8) (\frac{7}{5}q\Phi_3\Phi_8)</td>
</tr>
<tr>
<td>(F_4(q))</td>
<td>(d = 2)</td>
<td>(B_{2,\epsilon}) (B_2)</td>
<td>(\frac{1}{7}q^{14}\Phi_1^2\Phi_3\Phi_8) (\frac{7}{5}q\Phi_3\Phi_8)</td>
</tr>
<tr>
<td>(E_6(q))</td>
<td>(d = 1, 2)</td>
<td>(\phi_{20,2}) (\phi_{20,20})</td>
<td>(q^2\Phi_1\Phi_5\Phi_8\Phi_{12}) (q\Phi_1\Phi_5\Phi_8\Phi_{12})</td>
</tr>
<tr>
<td>(2E_6(q))</td>
<td>(d = 1, 2)</td>
<td>(\phi_{4,1}) (\phi_{4,13})</td>
<td>(q^6\Phi_1^2\Phi_8\Phi_{10}\Phi_{12}) (q^2\Phi_1\Phi_5\Phi_8\Phi_{10}\Phi_{12})</td>
</tr>
<tr>
<td>(E_7(q))</td>
<td>(d = 1, 2)</td>
<td>(\phi_{7,1}) (\phi_{7,46})</td>
<td>(q\Phi_{7}\Phi_{12}\Phi_{14}) (q\Phi_{12}\Phi_{14})</td>
</tr>
<tr>
<td>(E_8(q))</td>
<td>(d = 1, 2)</td>
<td>(\phi_{8,1}) (\phi_{35,2})</td>
<td>(q\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}) (q^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{30})</td>
</tr>
<tr>
<td>(2B_2(q), q^2 \equiv 2^{m+1})</td>
<td>Note: 3 \nmid</td>
<td>(2B_2[a]) (2B_2[b])</td>
<td>(\frac{1}{\sqrt{2}}q(q^2 - 1)) (\frac{q}{\sqrt{2}}(q^2 - 1))</td>
</tr>
<tr>
<td>(2F_4(q), q^2 \equiv 2^{m+1})</td>
<td>Note: 3 \mid (q^2 + 1)</td>
<td>cusp</td>
<td>(\frac{1}{7}q^4\Phi_1^2\Phi_5^2\Phi_8^2\Phi_6\Phi_{12}(\Phi_{24})^2) (\frac{1}{7}q^4\Phi_1^2\Phi_5^2\Phi_8^2\Phi_6^2(\Phi_{24})^2)</td>
</tr>
</tbody>
</table>

**Types** \(B_n\) and \(C_n\), \(n \geq 2\). When \(S\) is type \(B_n\) or \(C_n\) with \(n \geq 2\) defined in characteristic different than 3, Table 4 exhibits at least two distinct unipotent characters in \(\text{Irr}_3(B_0(S))^{\sigma_1}\) that are \(\text{Aut}(S)\)-invariant, with the exception of the case \(S = PSp_4(2^p)\) with \(p\) odd. In the latter situation, we may instead consider the characters indexed by \((1, 2)\) and \((0, 1)\) with degrees \(\frac{3}{2}(q^2 + 1)\) and \(\frac{3}{2}(q - 1)^2\), respectively. (Note that we do not consider \(PSp_4(2) \cong S_6\).) These characters lie in \(\text{Irr}_3(B_0(S))^{\sigma_1}\) and the latter character extends to \(\text{Aut}(S)\). (However, we remark that the first character is not \(\text{Aut}(S)\)-invariant, as in this case it is switched with \((0, 1)\) under the action of the graph automorphism, by [Mal08, Theorem 2.5]).

**Type** \(D_n\) and \(2D_n\), \(n \geq 4\). In this case, if \(S\) is not \(D_4(q)\), Tables 5 and 6 list at least two distinct unipotent characters that are \(\text{Aut}(S)\)-invariant members of
TABLE 3. Some Unipotent Characters in $\text{Irr}_{3'}(B_0(S))^n_1$ for type $A_{n-1}^e(q)$ with $n \geq 4$ and $3 \nmid q$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Additional Condition on $n, e$</th>
<th>Partition</th>
<th>$\chi(1)_{q^e}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \geq 6$</td>
<td>$e = 2$ and $n$ even; or $e = 1$ and $3 \nmid (n-1)$</td>
<td>$(1, n-1)$</td>
<td>$q^{n-1-e^{n-1}}$</td>
</tr>
<tr>
<td></td>
<td>$e = 1$ and $3 \nmid (n-1)$</td>
<td>$(1^{n-2}, 2)$</td>
<td>$q^{-e}$</td>
</tr>
<tr>
<td>$n \geq 6$</td>
<td>$e</td>
<td>n$ and $3 \nmid n$</td>
<td>$(2, n-2)$</td>
</tr>
<tr>
<td></td>
<td>$e = 2$ and $n$ odd and $3 \mid (n-2)$</td>
<td>$(1^{n-3}, 3)$</td>
<td>$(q^{-e})(q^{n-2-e^{n-2}})$</td>
</tr>
<tr>
<td>$n \geq 6$</td>
<td>$e = 2$ and $n$ odd and $3 \mid (n-1)$</td>
<td>$(1, 1, 2, n-4)$</td>
<td>$(q^{-e})(q^{n-2-e^{n-3}})(q^{n-1-e^{n-1}})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2, n-2)$</td>
<td>$(q^{-e})(q^{n-1-e^{n-1}})$</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$e = 1$</td>
<td>$(1, 4)$</td>
<td>$(q + e)(q^{2} + 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1, 1, 1, 2)$</td>
<td></td>
</tr>
<tr>
<td>$n = 5$</td>
<td>$e = 2$</td>
<td>$(2, 3)$</td>
<td>$q^{2} - e$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1, 1, 3)$</td>
<td>$(q^{2} + 1)(q^{2} + eq + 1)$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$e = 2$</td>
<td>$(1, 3)$</td>
<td>$q^{2} + eq + 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1, 1, 2)$</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4. Some Unipotent Characters in $\text{Irr}_{3'}(B_0(S))^n_1$ for type $B_n(q)$ and $C_n(q)$ with $n \geq 2$ and $3 \nmid q$

<table>
<thead>
<tr>
<th>Conditions on $q, n$</th>
<th>Symbol</th>
<th>$\chi(1)_{q^e}$ (possibly excluding factors of $1/2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \mid (q-1)$ or $3 \mid (q+1); n$ even; $3 \nmid (n-1)$ or $3 \mid (q+1); n$ odd; $3 \nmid n$</td>
<td>$\binom{0}{1}$</td>
<td>$(q^{-1})(q^{n-1}+1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (q-1); 3 \nmid n$</td>
<td>$\binom{0}{2}$</td>
<td>$(q^{-1})(q^{2n-3}+1)(q^{n-1}+1)$</td>
</tr>
<tr>
<td>$3 \mid (q-1); 3 \nmid (n-1)$ or $3 \mid (q+1); n$ even</td>
<td>$\binom{1}{0}$</td>
<td>$(q^{n-1}-1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (q-1); 3 \nmid n$</td>
<td>$\binom{0}{n-1}$</td>
<td>$(q^{n-1}-1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (n-1)$</td>
<td>$\binom{1}{n-1}$</td>
<td>$(q^{n-1}-1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (q+1); n$ odd or $3 \mid (q+1); 3 \nmid n$</td>
<td>$\binom{0}{n}$</td>
<td>$(q^{n-1}+1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (q-1); 3 \nmid n$</td>
<td>$\binom{1}{n}$</td>
<td>$(q^{n-1}+1)(q^{n+1})$</td>
</tr>
<tr>
<td>$3 \mid (q+1); n$ odd or $3 \mid (q+1); 3 \nmid n$</td>
<td>$\binom{0}{n-2}$</td>
<td>$(q^{2n-1}-1)(q^{n-1}+1)$</td>
</tr>
</tbody>
</table>

$\text{Irr}_{3'}(B_0(S))^n_1$. If $S$ is $D_4(q)$ and $3 \mid (q-1)$, we may instead take the unipotent characters labeled by symbols $\binom{1}{0} 3$ and $\binom{3}{1}$ with $\chi(1)_{q^e} = \frac{1}{2}(q + 1)^3(q^{2} + 1)^2$, respectively. When $3 \mid (q+1)$, we may take the characters index by $\binom{3}{1}$ and $\binom{0}{1} 2 3$, the latter of which satisfies $\chi(1)_{q^e} = \frac{1}{2}(q - 1)^3(q^{2} - 1)$. □
### Table 5. Some Unipotent Characters in \( \text{Irr}_3(B_0(S))^{\sigma_1} \) for type \( D_n(q) \) with \( n \geq 5 \) and \( 3 \nmid q \)

<table>
<thead>
<tr>
<th>Conditions on ( q, n )</th>
<th>Symbol</th>
<th>( \chi(1)q' ) (possibly excluding factors of 1/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \nmid (n-1) )</td>
<td>( \binom{1}{n} )</td>
<td>( \frac{q^{2(n-1)-1}}{q^2 - 1} )</td>
</tr>
<tr>
<td>( 3 \mid (n-2) ) or ( 3 \mid (q+1); 3 \mid n; n \text{ odd} )</td>
<td>( \binom{n-2}{2} )</td>
<td>( \frac{(q^{2(n-1)-1})(q^{n-4}+1)}{(q^2-1)^2(q^2+1)} )</td>
</tr>
<tr>
<td>( 3 \mid (q-1); 3 \mid (n-1) ) or ( 3 \mid (q+1); 3 \mid (n-2); n \text{ even or } 3 \mid (q+1); 3 \mid n; n \text{ odd} )</td>
<td>( \binom{0}{1} n )</td>
<td>( \frac{(q^{n-1})(q^{n-2}+1)(q^{n-3}+1)}{(q^2-1)^2} )</td>
</tr>
</tbody>
</table>

### Table 6. Some Unipotent Characters in \( \text{Irr}_3(B_0(S))^{\sigma_1} \) for type \( ^2D_n(q) \) with \( n \geq 4 \) and \( 3 \nmid q \)

<table>
<thead>
<tr>
<th>Conditions on ( q, n )</th>
<th>Symbol</th>
<th>( \chi(1)q' ) (possibly excluding factors of 1/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \nmid (n-1) )</td>
<td>( \binom{0}{1} n )</td>
<td>( \frac{q^{2(n-1)-1}}{q^2 - 1} )</td>
</tr>
<tr>
<td>( 3 \mid (n-1) ) or ( 3 \mid (q-1); 3 \mid n \text{ or } 3 \mid (q+1); 3 \mid n; n \text{ even or } 3 \mid (q+1); 3 \mid (n-2); n \text{ odd} )</td>
<td>( \binom{1}{n-1} \emptyset )</td>
<td>( \frac{(q^{n+1})(q^{n-2}-1)}{q^2 - 1} )</td>
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<td>( 3 \mid (q-1); 3 \mid (n-1) ) or ( 3 \mid (q+1); 3 \mid (n-2); n \text{ odd or } 3 \mid (q+1); 3 \mid n; n \text{ even} )</td>
<td>( \binom{2}{n-2} \emptyset )</td>
<td>( \frac{(q^{2(n-1)-1})(q^{n+1})(q^{n-4}+1)}{(q^2-1)^2(q^2+1)} )</td>
</tr>
<tr>
<td>( 3 \mid n )</td>
<td>( \binom{0}{1} \frac{2}{n} )</td>
<td>( \frac{(q^{2(n-1)-1})(q^{n-2}-1)}{(q^2-1)^2(q^2+1)} )</td>
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<tr>
<td>( 3 \mid (q-1); 3 \mid (n-1) ) or ( 3 \mid (q+1); 3 \mid n \text{ odd or } 3 \mid (q+1); 3 \mid (n-2); n \text{ even} )</td>
<td>( \binom{1}{2} n \emptyset )</td>
<td>( \frac{(q^{n+1})(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2} )</td>
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<tr>
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<td>( \binom{0}{1} \frac{n-1}{2} )</td>
<td>( \frac{(q^{n+1})(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2} )</td>
</tr>
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</table>
We next establish Theorem 3.7 for the case that \( S \) has cyclic Sylow 3-subgroups, which we recall from Proposition 3.6 occurs when \( S = PSL_2(q) \) for \( 3 \mid q \) and when \( S = PSL_3(q) \) for \( 3 \mid (q + \epsilon) \).

**Proposition 3.10.** Let \( S = PSL_2(q) \) with \( 3 \mid q \) or \( PSL_3(q) \) with \( 3 \mid (q + \epsilon) \). Then there exist nontrivial \( \chi_1, \chi_2 \in \text{Irr}_3(B_0(S))^{\sigma_1} \) such that \( \chi_1 \) extends to \( \text{Aut}(S) \).

**Proof.** First let \( S = PSL_2(q) \) with \( 3 \mid q \). In this case, every character of \( S \) is either 3-defect zero or has degree prime to 3. As before, the Steinberg character is a member of \( \text{Irr}_3(B_0(S))^{\sigma_1} \) and extends to \( \text{Aut}(S) \). Further, the only two unipotent characters, \( 1_{\tilde{G}} \) and \( \text{St}_{\tilde{G}} \), both lie in the principal block of \( \tilde{G} = GL_2(q) \), and hence there is a unique unipotent block of \( \tilde{G} \). We may take \( \chi_1 = \text{St}_S \) as before.

Now, let \( s \in \tilde{G} = GL_2(q) \) have eigenvalues \( \zeta, \zeta^{-1} \), where \( \zeta \in \mathbb{F}_q^\times \) has order 3. Then the semisimple character \( \chi_s \in \text{E} = E(G, s) \subseteq E_3(\tilde{G}, 1) \) lies in the principal block of \( \tilde{G} \) and is trivial on \( Z(\tilde{G}) \) since \( s \in SL_2(q) \cong \tilde{G}/1 \). Since \( s \) is not conjugate to \( s \) for \( 1 \neq z \in Z(\tilde{G}) \), we also see \( \chi_s \) is irreducible on restriction to \( G \). Further, Lemma 3.1 yields that \( \chi_s \) is fixed by \( \sigma_1 \). Then the restriction \( (\chi_s)_G \) lies in \( B_0(G) \) since the principal block of \( \tilde{G} \) covers a unique block of \( G \). Finally, in this case \( \chi_s(1) = q + \eta \), where \( \eta \in \{ \pm 1 \} \) is such that \( 3 \mid q - \eta \). Hence this character may be viewed as a member of \( \text{Irr}_3(B_0(S))^{\sigma_1} \), arguing as before.

Now let \( S = PSL_3(q) \) with \( 3 \mid (q + \epsilon) \). Then \( S = G = SL_3(q) \) and \( \tilde{G} = G \times Z(\tilde{G}) \). Since the unipotent characters of \( G \) are \( 1_G, \text{St}_G \), and a character of degree \( q(q + \epsilon) \), we see that again \( B_0(G) \) is the only unipotent block of maximal defect (as the other has defect zero). Then every character of \( E_3(G, 1) \) with \( 3' \)-degree is a member of \( B_0(G) \). We may again take \( \chi_1 = \text{St}_G \). Taking \( s \in \tilde{G}^\ast \) to have eigenvalues \( \{ \zeta, \zeta^{-1}, 1 \} \), where \( \zeta \in \mathbb{F}_q^\times \) has order 3, the corresponding character of \( G \) has degree \( q^3 - \epsilon \), and we may again view \( (\chi_s)_G \) as a character of \( \text{Irr}_3(B_0(S))^{\sigma_1} \). \( \square \)

**Proposition 3.11.** Let \( S = PSL_3(q) \) with \( 3 \mid (q - \epsilon) \). Then there exist nontrivial \( \chi_1, \chi_2, \chi_3 \in \text{Irr}_3(B_0(S))^{\sigma_1} \) such that \( \chi_1 \) and \( \chi_2 \) extend to \( \text{Aut}(S) \).

**Proof.** In this case, we see that all three unipotent characters are members of \( \text{Irr}_3(\tilde{G}) \) and that there is a unique unipotent block \( B_0(\tilde{G}) \). Further, the unipotent characters are rational-valued, and therefore are members of \( \text{Irr}_3(B_0(\tilde{G}))^{\sigma_1} \). Then we may take \( \chi_1 \) and \( \chi_2 \) to be the restrictions to \( G \) (viewed as a character of \( S \)) of the two nontrivial unipotent characters.

The semisimple element \( s \in \tilde{G}^{\ast} \) with eigenvalues \( \{ \zeta, \zeta^{-1}, 1 \} \), where \( \zeta \in \mathbb{F}_q^\times \) has order 3, is now conjugate to \( sz \) where \( z = \zeta \cdot I_3 \in Z(\tilde{G}^{\ast}) \). The corresponding semisimple character has degree \( \chi_s(1) = (q + \epsilon)(q^2 + eq + 1) \), so \( \chi_s \in \text{Irr}(B_0(\tilde{G}))^{\sigma_1} \) satisfies \( 3 \mid \chi_s(1) \) and is not irreducible on restriction to \( G \). Then the constituents of the restriction to \( G \) are members of \( \text{Irr}_3(B_0(G)) \), and are trivial on \( Z(G) \) since \( s \in \tilde{G}^{\ast} \). So it
suffices to see that they are also $\sigma_1$-invariant, using the character table available in CHEVIE.

Together, Propositions 3.9 through 3.11 yield Theorem 3.7 for the simple groups of Lie type in non-defining characteristic, completing parts (i) and (ii).

3.3.2. Lie Type in Defining Characteristic for $p = 3$. We now consider the case $S$ is as in Section 3.1.1 with $G$ of simply connected type defined in characteristic 3.

Let $(G^*, F^*)$ be dual to $(G, F)$. Keep in mind the notations and considerations of Section 3.1, where now $q$ is a power of 3. Note that $|\hat{S}/S| = |Z(G)|$, and this is 1 unless $G$ is of classical type or $G = E_{7,sc}(q)$.

Since $\hat{G}/G$ has size prime to 3, it follows that any irreducible character of $G$ lying under $\text{Irr}_3'(\hat{G})$ is a member of $\text{Irr}_3'(G)$. Since $|\sigma_1|$ is a power of 3, we further see that for any $\hat{\chi} \in \text{Irr}_3'(\hat{G})^{\sigma_1}$, there is a member of $\text{Irr}_3'(G)^{\sigma_1}$ lying under $\hat{\chi}$. We also have $\text{Irr}_3'(B_0(S)) = \text{Irr}_3'(S)$, so any member of $\text{Irr}_3'(G)$ with $Z(G)$ in its kernel may be viewed as a member of $\text{Irr}_3'(B_0(S))$.

Now, given a semisimple element $s \in \hat{G}^*$, we have $|s|$ is prime to 3, and hence the Lusztig series $E(\hat{G}, s)$ is fixed by $\sigma_1$ using Lemma 3.1. Then in particular, the unique semisimple character $\hat{\chi}_s \in \text{Irr}_3'(\hat{G})$ in this series must be fixed by $\sigma_1$.

Hence to illustrate three characters of $\text{Irr}_3'(G)^{\sigma_1}$ that are not $\text{Aut}(S)$-conjugate, it suffices to show that there are semisimple elements $s_1, s_2, s_3 \in \hat{G}^* $ such that

1. $s_1, s_2, s_3$ are not pairwise conjugate under any graph-field automorphism and
2. $s_i$ is not $\hat{G}^*$-conjugate to $s_j z$ for $i \neq j$ and $z \in Z(\hat{G}^*)$.

In most cases, we further ensure that one of these characters is $\text{Aut}(S)$-invariant, by choosing $s_1$ so that

3. the class of $s_1$ is invariant under graph-field automorphisms and
4. $s_1$ is not $\hat{G}^*$-conjugate to $s_1 z$ for any $1 \neq z \in Z(\hat{G}^*)$.

Property (3) will ensure that $\hat{\chi}_{s_1}$ is invariant under graph-field automorphisms, using [NTT08, Corollary 2.4], and property (4) will imply that $\hat{\chi}_{s_1}$ restricts irreducibly to $G$, so the resulting character of $G$ is $\text{Aut}(S)$- and $\sigma_1$-invariant. Finally, we will choose $s_1, s_2, s_3$ and $s_3$ such that

5. $s_i \in [\hat{G}^*, \hat{G}^*]$ for $i = 1, 2, 3$.

so that the $\hat{\chi}_{s_i}$ are trivial on $Z(\hat{G})$, ensuring that all three characters of $G$ may be viewed as characters of $\text{Irr}_3'(B_0(S))^{\sigma_1}$ from the above discussion.

**Proposition 3.12.** Let $S = G_2(q), 3D_4(q), 3G_2(q), E_6^7(q), E_7(q), F_4(q)$ or $E_8(q)$ be simple with $q$ a power of 3. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \text{Irr}_3'(B_0(S))^{\sigma_1}$ that are pairwise not $\text{Aut}(S)$-conjugate and such that $\chi_1$ is $\text{Aut}(S)$-invariant.

**Proof.** Note that we may assume $S$ is not one of the groups from Proposition 3.8. The character degrees in these cases are available at [Lüb07].
If $S$ is $G_2(q)$, $3D_4(q)$, $2G_2(q)$, $E_6(q)$, $2E_6(q)$, $F_4(q)$ or $E_8(q)$, then $\tilde{G} = G = S$ and there is a unique character of degree $(q^4 + q^2 + 1)$, $(q^8 + q^4 + 1)$, $(q^4 - q^2 + 1)$, $(q^8 + q^4 + 1)(q^2 + 1)(q^6 + q^3 + 1)$, $(q^8 + q^4 + 1)(q^4 + 1)(q^6 - q^3 + 1)$, $(q^8 + q^4 + 1)$, or $\Phi_3\Phi_5\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$, respectively, which therefore must be $\sigma_1$- and $\text{Aut}(S)$-invariant. Similarly, $E_7(q)$ has a unique character of degree $\Phi_3\Phi_6\Phi_7\Phi_{12}\Phi_{14}\Phi_{18}$, which restricts irreducibly from a character of $\tilde{S} = E_7(q)_{\text{ad}}$.

Finally, in each case there are at least two more semisimple characters with different degrees, which must yield members of $\text{Irr}_3(B_0(S))^{\sigma_1}$ by the above discussion. □

**Proposition 3.13.** Let $S = \text{PSL}_n^e(q)$ be simple with $q$ a power of 3 and $n \geq 2$ and let $X \leq \text{Aut}(S)$ such that $X/S$ is a Sylow 3-subgroup of $\text{Aut}(S)/S$. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \text{Irr}_3(B_0(S))^{\sigma_1}$ that are pairwise not $\text{Aut}(S)$-conjugate and such that $\chi_1$ is $X$-invariant. Further, if $n \geq 3$, then $\chi_1$ may be chosen to be $\text{Aut}(S)$-invariant, and if $n \geq 5$, then $\chi_1, \chi_2, \chi_3$ may all be chosen to be $\text{Aut}(S)$-invariant.

**Proof.** Throughout, let $\delta \in \mathbb{F}_q^\times$ have order 4 and assume $S$ is not isomorphic to one of the groups in Proposition 3.8. Recall that the conjugacy classes of semisimple elements in $\tilde{G}^* = GL_2(q)$ are determined by their eigenvalues and that $\mathbb{Z}(\tilde{G}^*)$ is comprised of scalar matrices.

If $n = 2$, $[\tilde{S}/S] = 2$ and $\text{Aut}(S)/\tilde{S}$ is generated by a field automorphism. The semisimple elements $s_1, s_2,$ and $s_3$ with eigenvalues $\{\delta, \delta^{-1}\}$, $\{\xi, \xi^{-1}\}$, and $\{\zeta, \zeta^{-1}\}$ with $\xi, \zeta \in \mathbb{F}_q^\times$ and $|\xi| \neq 4 \neq |\zeta|$ satisfy properties (1), (2), (3), and (5). Now, since $\tilde{\chi}_{s_1}$ is fixed by field automorphisms, and hence by $X$, and since $[\tilde{S}/S]$ is relatively prime to 3, we see that the irreducible constituents of the restriction $(\tilde{\chi}_{s_1})_G$ are still fixed by $X$ and by $\sigma_1$.

If $n = 3$ or 4, let $s_1, s_2, s_3$ have eigenvalues $\{\delta, \delta^{-1}\}$, $\{-1, -1\}$, and $\{\xi, \xi^{-1}\}$ with remaining eigenvalues 1, with $|\xi| > 2$ dividing $q + \eta$ if $4|q - \eta$. Then these satisfy (1)-(5).

Now suppose that $n \geq 5$. Consider semisimple elements $s_1, s_2,$ and $s_3$ of $\tilde{G}^* = GL_n(q)$ with eigenvalues $(\delta, \delta^{-1}, 1, \ldots, 1)$, $(1, -1, -1, \ldots, 1)$, and $(\delta, \delta^{-1}, \delta, \delta^{-1}, 1, \ldots, 1)$, respectively. If $n = 6$, instead define $s_3$ to have eigenvalues $(-1, -1, -1, -1, 1, 1)$. Then these satisfy (1)-(5), and in fact properties (3) and (4) are held by all three elements. Hence the corresponding semisimple characters $\tilde{\chi}_{s_i}$ of $\tilde{G}$ are invariant under graph-field automorphisms and restrict irreducibly to members of $\text{Irr}_3(B_0(G))^{\sigma_1}$ that are trivial on $\mathbb{Z}(G)$. Hence these restrictions are members of $\text{Irr}_3(B_0(S))^{\sigma_1}$ invariant under $\text{Aut}(S)$. □

**Proposition 3.14.** Let $q$ be a power of 3. Let $S = \text{PSp}_{2n}(q), \text{PO}_{2n+1}(q)$, or $\text{PO}_{2n}^e(q)$ be simple with $n \geq 2, 3, 4$ respectively. Then there exist nontrivial $\chi_1, \chi_2, \chi_3 \in \text{Irr}_3(B_0(S))^{\sigma_1}$ that are pairwise not $\text{Aut}(S)$-conjugate and such that $\chi_1$ is invariant under $\text{Aut}(S)$.
Proof. We may again assume $S$ is not one of the groups in Proposition 3.8. Let $\delta \in \mathbb{F}_q^\times$ with $|\delta| = 4$, and we keep the notation from Section 3.1. Let $\Phi$ and $\Delta := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a system of roots and simple roots, respectively, for $\tilde{G}^*$ with respect to a maximal torus $\tilde{T}^*$, following the standard model described in [GLS03, Remark 1.8.8]. Then $\Phi$ is type $B_n$, $C_n$, or $D_n$ in the case $S = PS\ell_2(q), PO_{2n+1}(q)$, or $PO_{2n}(q)$, respectively. Further, $\Phi$ has no nontrivial graph automorphism unless we are in the case of $D_n$, in which case all members of $\Delta$ have the same length and that automorphism has order 2 unless $n = 4$.

We use the notation as in [GLS03] for the Chevalley generators. In particular, given $\alpha \in \Phi$, let $h_\alpha$ denote the corresponding coroot. Let $K := [\tilde{G}^*, \tilde{G}^*]$, so we have $h_\alpha(t) \in K$ for $t \in \mathbb{F}_q^\times$ by [GLS03, Theorem 1.10.1(a)] and $\tilde{G}^* = K \cdot Z(\tilde{G}^*)$. Notice that for $s, s' \in K$ (not necessarily distinct), we have $s$ is $\tilde{G}^*$-conjugate to $s'z$ for $z \in Z(\tilde{G}^*)$ if and only if $z \in Z(K)$ and the conjugating element can be chosen in $K$.

By [GLS03, Theorem 1.12.4] and [CE04, 15.1], $K$ is isomorphic as an abstract group to the simply connected simple algebraic group $(\tilde{G}^*)_{sc}$ associated to $\tilde{G}^*$, and the Chevalley relations and generators of $(\tilde{G}^*)_{sc}$ and $K$ may be identified. We will make this identification. In particular, choosing $s_1, s_2$, and $s_3$ in $K$, the properties (1)-(4) may be verified by computation in $K$ rather than $\tilde{G}^*$.

Let $T$ denote a maximal torus of $K$ under this identification, and note that $T = \langle h_\alpha(t) \mid t \in \mathbb{F}_q^\times, \alpha \in \Phi \rangle$, and $N_K(T) = \langle T, n_\alpha(1) \mid \alpha \in \Phi \rangle$. Further, note that $W := N_{\tilde{G}^*}(\tilde{T}^*)/\tilde{T}^* \cong N_K(T)/T$. By [DM91, Cor. 0.12], we know that $N_K(T)$ controls fusion in $T$, so two elements of $T$ are conjugate if and only if there is a conjugating element in $W$. Further, we have an isomorphism $(\mathbb{F}_q^\times)^n \to T$ given by $(t_1, \ldots, t_n) \mapsto \prod_{i=1}^n h_{\alpha_i}(t_i)$.

Now using the standard model for $\Phi$ and $\Delta$ as in [GLS03], since $\Phi$ is type $B_n, C_n$, or $D_n$, we have $\alpha_i := e_i - e_{i+1}$ for $1 \leq i \leq n - 1$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for the $n$-dimensional Euclidean space. Here $W \leq C_2 \wr S_n$, where the generators of the base subgroup $C_2^n$ act via negation on the $e_i$'s and the copy of $S_n$ permutes the $e_i$'s.

Using this information and the description of $Z(K)$ in [GLS03, Table 1.12.6], computation with the Chevalley relations yields that the element $s'_1 := h_{\alpha_1}(\delta)$ is not $\tilde{G}^*$-conjugate to $s'_1 z$ for any $1 \neq z \in Z(\tilde{G}^*)$. If $\delta \in \mathbb{F}_q^\times$, we see that $s'_1$ is $F^*$-fixed, and we write $s_1 := s'_1$. Otherwise, let $s_{\alpha_1} \in W$ induce the reflection corresponding to $\alpha_1$. Then $s_1 := s_{\alpha_1}^g$ is $F^*$-fixed, where $g \in \tilde{G}^*$ satisfies $g^{-1}F^*(g) = s_{\alpha_1}$. (Note that such a $g$ exists by the Lang-Steinberg theorem.)

Let $F_t$ denote a generating field automorphism such that $F_t(h_\alpha(t)) = h_\alpha(t^3)$ for $\alpha \in \Phi$ and $t \in \mathbb{F}_q^\times$. Then $s'_1$ is $\tilde{G}^*$-conjugate to $F_t(s'_1)$, taking for example $s_{\alpha_1}$ as the conjugating element. Hence $s_1$ is also $\tilde{G}^*$-conjugate to $F_t(s_1)$. Since the $C_{G^*}(t^*(s_1))$
is connected, using \([\text{Bon05, Corollary 2.8(a)}]\), this yields that the \(\tilde{G}^s\)-conjugacy class of \(s_1\) is fixed by field automorphisms, using \([\text{DM91, (3.25)}]\). Further, by construction, the \(\tilde{G}^s\)-conjugacy class of \(s_1\) is fixed by graph automorphisms unless \(\Phi\) is type \(D_4\). In the latter case, we may make similar considerations using \(s_1' := h_{\alpha_2}(\delta)\).

Now, further taking \(s_2 := h_{\alpha_1}(-1)\) and \(s_3 \in K^{F^s}\) an element of order larger than 4, we obtain properties (1)-(5).

Theorem 3.7 now follows from Propositions 3.8 – 3.14, completing the proof of Theorem A.

**References**


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